CELESTIAL MECHANICS

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ABSTRACT

Mathematical relationships can be used to describe both the motion of the planets and the positioning of celestial bodies. Kepler's Laws, formulated at the beginning of the seventeenth century by German mathematician Johannes Kepler, explain the nature of planets' motion around the sun. In this paper, we prove these laws and derive a series of mathematical equations that illustrate the geometric relationships of an elliptical orbit using a combination of spherical trigonometric principles and calculus. We then utilize these relationships to calculate the time of sunrise and sunset from any given point on the Earth.

INTRODUCTION

The planets in our solar system move in elliptical orbits around the sun. The motion of these celestial bodies can be modeled using Kepler's Laws, elliptical geometry, and spherical trigonometry. Elliptical geometry allows us to utilize a heliocentric model, with the sun at one focus, to trace celestial motion. By deriving Kepler's Laws, we confirm that a planet travels in an elliptical path, sweeps out equal areas in equal times, and has a period whose square is directly proportional to the cube of the length of its semi-major axis. Furthermore, through spherical trigonometry, a model of these heavenly bodies can be produced which simplifies their motions and can be used to predict the times of sunrise and sunset. The formulae derived to aid the observation of celestial motion can be applied to the Earth or to other planets, thus providing a general prototype with which to calculate planetary location.

ELLIPTICAL GEOMETRY

The Ellipse

We will see that each planet moves in an elliptical orbit around the sun. The equation of an ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let

$$c = \sqrt{a^2 - b^2} \tag{1}$$

so that $(\pm c, 0)$ represent the foci of the ellipse.

This elliptical orbit has a set eccentricity, which indicates how closely it resembles a circle. While an ellipse with a small eccentricity will tend to take a circular shape, an ellipse with a large eccentricity will appear flattened. This causes a great difference between its *a* and *b* values. Eccentricity is the ratio between *c*, the distance between the center and the foci, and *a*, the length of the semi-major axis; it is defined by:

$$e = \frac{c}{a}$$

The sun's position is at a focus of the ellipse, making it convenient to shift the graph of the ellipse so that the sun is at the origin. To accomplish this, we need to shift the graph to the left by a distance *c*. To aid in tracing planetary orbits it is helpful to put the equation of the ellipse into polar form. This yields:

$$r^2(1 - e^2\cos^2\theta) + r[2ae\cos\theta(1 - e^2)] + a^2(1 - e^2)^2 = 0$$

This formula can be plugged into the quadratic formula to solve for *r.* Algebraic manipulation of this formula yields:

$$r = \frac{a(e^2 - 1)(e\cos\theta \pm 1)}{(1 - e\cos\theta)(1 + e\cos\theta)}$$

The positive sign in the above equation should be chosen because the negative sign would produce negative values of *r,* which are impossible. For this reason the $(1 - e\cos\theta)$ term is dropped to give the simplified version of the equation:

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{2}$$

Primarily, the planets have elliptical orbits that resemble circles (small eccentricities). For this reason, it is convenient to position a circle with radius *a* and the same center as the ellipse about a given elliptical orbit. The planet's position at a given time can be easily projected onto the circle. (Fig.1)

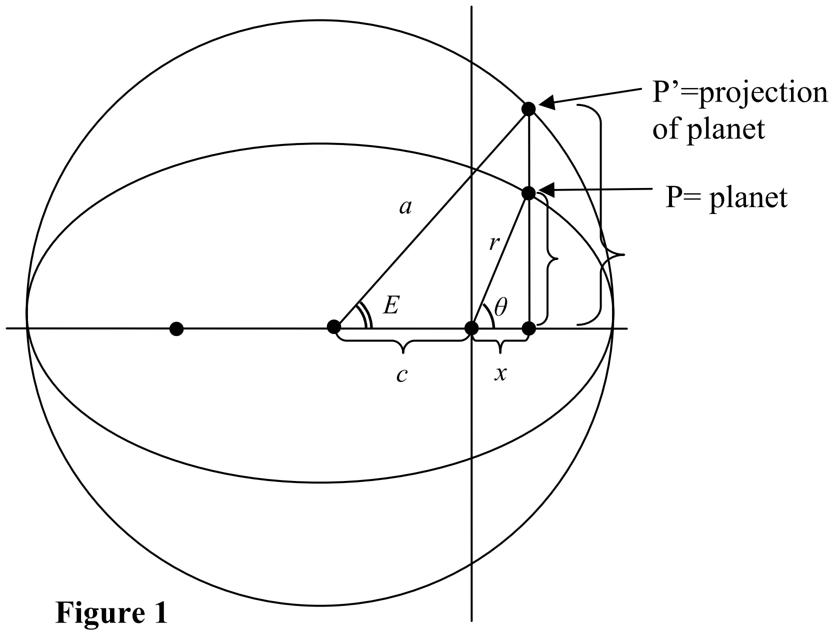

Figure 1

In order to plot the planet's location on the ellipse at a given time, we must derive a relationship between angle E and angle θ.

<u>The relationship between E and θ</u>

Given the diagram, we find the following relationships:

$$\cos E = \frac{x+c}{a}$$

$$\cos\theta = \frac{x}{r}$$

By substituting terms, we obtain new values for both angles involving variables a and e only.

$$\cos E = \frac{r\cos\theta + ea}{a}$$

$$\cos\theta = \frac{a\cos E - ea}{r}$$

By substituting the earlier definition of r in the equation for $\cos\theta$, we find that:

$$\cos\theta = \frac{(\cos E - e)(1+e\cos\theta)}{(1-e^2)}$$

By rearranging terms, cos$E$ can be consolidated on one side so that:

$$\cos E = \frac{(1-e^2)\cos\theta}{(1+e\cos\theta)} + e$$

After algebraic manipulation, we are left with the final equation for the relationship between the two angles:

$$\cos E = \frac{(\cos\theta + e)}{1+e\cos\theta} \tag{3}$$

To simplify the equation relating the angle θ and angle E, we use half-angle formulas and solve to find a relationship between $\tan\frac{E}{2}$ and $\tan\frac{\theta}{2}$. The half-angle formula of any angle E is given by the equation:

$$\tan\frac{E}{2} = \sqrt{\frac{(1-\cos E)}{(1+\cos E)}}$$

Substituting equation (3) for cos$E$ yields:

$$\tan\frac{E}{2} = \sqrt{\frac{(e-1)(\cos\theta-1)}{(e+1)(\cos\theta+1)}}$$

or

$$\tan\frac{E}{2} = \sqrt{\frac{(1-e)}{(1+e)}}\tan\frac{\theta}{2} \tag{4}$$

This relationship allows the simple conversion between E and θ.

We can now use this information to derive Kepler's Laws of Planetary Motion.

KEPLER'S LAWS OF PLANETARY MOTION

Kepler's First Law

To begin the derivation of Kepler's First Law, we begin with two force functions:

$$\boldsymbol{F} = m\boldsymbol{a} = m\frac{d^2\boldsymbol{s}}{dt^2}$$

and

$$\boldsymbol{F} = -\frac{GMm}{\boldsymbol{r}^2}$$

where m is the mass of the planet, M is the mass of the sun, G is the universal gravitational constant, and r is the distance from the sun. We now equate them and break the force vectors into separate x and y components. We then change to polar coordinates by eliminating x and y and expressing force in terms of r and θ. The equations now become:

$$\boldsymbol{F_x} = -\frac{GMm}{\mathrm{r}^2}\cos\theta = m\frac{d^2}{dt^2}(r\cos\theta)$$
$$\boldsymbol{F_y} = -\frac{GMm}{\mathrm{r}^2}\sin\theta = m\frac{d^2}{dt^2}(r\sin\theta)$$

Differentiating, we have:

$$\cos\theta\frac{d^2r}{dt^2} - 2\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} - r\sin\theta\frac{d^2\theta}{dt^2} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2}\cos\theta \tag{5}$$

and

$$\sin\theta\frac{d^2r}{dt^2} + 2\cos\theta\frac{dr}{dt}\frac{d\theta}{dt} + r\cos\theta\frac{d^2\theta}{dt^2} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2}\sin\theta \tag{6}$$

We will now form two new equations. For the first, we multiply equation (5) by $-\sin\theta$ and equation (6) by $\cos\theta$, then add the two equations to obtain:

$$2\frac{dr}{dt}\frac{d\theta}{dt} - r\frac{d^2\theta}{dt^2} = 0 \tag{7}$$

For the second equation, we multiply equation (5) by $\cos\theta$ and equation (6) by $\sin\theta$, then add them to obtain:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \tag{8}$$

Next, we introduce a new variable $p = \dfrac{d\theta}{dt}$. Equation (7) becomes:

$$\frac{2}{r}\frac{dr}{dt} = -\frac{1}{p}\frac{dp}{dt}$$

After integrating both sides, we obtain:

$$2\ln r = -\ln p + c$$

which becomes:

$$r^2 = \frac{e^c}{p} \qquad \text{or} \qquad pr^2 = h$$

where $h = e^c$. Back-substituting p, this becomes:

$$r^2 \frac{d\theta}{dt} = h \qquad \text{or} \qquad \frac{d\theta}{dt} = \frac{h}{r^2} \tag{9}$$

We will now substitute this into equation (8), yielding:

$$\frac{d^2 r}{dt^2} - \frac{h^2}{r^3} = -\frac{GM}{r^2} \tag{10}$$

At this point, we eliminate the variable r by introducing the variable $u = 1/r$. By the chain rule,

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}$$

and

$$\frac{d^2 r}{dt^2} = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{h}{r^2} = h^2 u^2 \frac{d^2 u}{d\theta^2}$$

Now, substituting this into equation (10), we obtain:

$$\frac{d^2 u}{d\theta^2} + u = \frac{GM}{h^2}$$

Clearly, the u must be the cosine function:

$$u = B\cos(\theta + C) + \frac{GM}{h^2}$$

We now back-substitute in the variable r, yielding:

$$r = \frac{\dfrac{h^2}{GM}}{1 + \dfrac{Bh^2}{GM}\cos(\theta + C)} \tag{11}$$

We assume that when $\theta = 0$, the planet is at its perihelion, i.e. the distance r from the sun is minimized. Thus, C must also equal 0. The equation now becomes:

$$r = \frac{\dfrac{h^2}{GM}}{1 + \dfrac{Bh^2}{GM}\cos\theta}$$

If we let

$$e = \frac{Bh^2}{GM} \tag{12}$$

Then the equation becomes:

$$r = \frac{e}{B(1 + e\cos\theta)} \tag{13}$$

Now we must eliminate the variable B. To do this, we begin with equation (13). When angle $\theta = 0°$, the planet is at perihelion and is a distance r_p from the sun. When $\theta = 180°$, the planet is at aphelion and is a distance r_a from the sun. The sum of these two distances is equal to the distance across the semi-major axis, i.e., $2a$ (Fig. 2). Therefore,

$$\frac{e}{B(1 + e\cos 0°)} + \frac{e}{B(1 + e\cos 180°)} = 2a$$

or

$$\frac{e}{B(1 + e)} + \frac{e}{B(1 - e)} = 2a$$

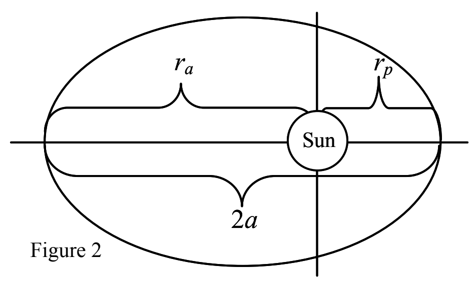

Figure 2

Solving for B, we have:

$$B = \frac{e}{a(1 - e^2)} \tag{14}$$

Now if we substitute equation (14) into equation (13), we have:

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{15}$$

This is equivalent to equation (3), which has been proved previously. Therefore, planets trace elliptical paths as they rotate around the sun.

Kepler's Second Law

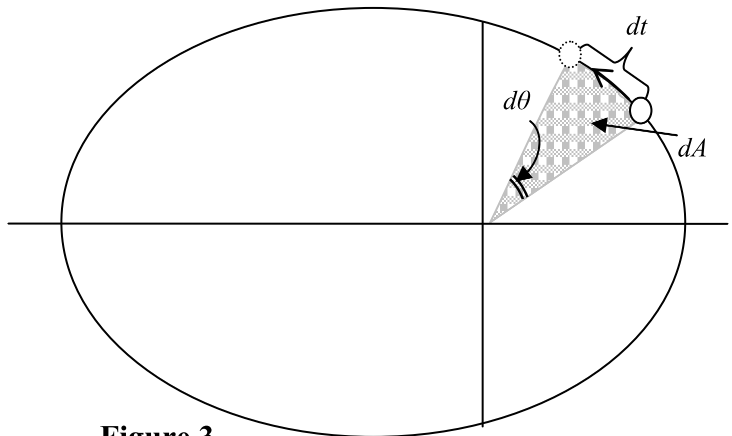

Figure 3

We begin this proof by stating that as the planet moves, it sweeps out an area inside the elliptical orbit. As θ increases infinitesimally, i.e. $d\theta$, it sweeps out an infinitesimally small area within the ellipse, i.e. dA (Fig. 3). With such small dimensions, the ellipse takes on the characteristics of a circle, and dA is roughly equal to a sector of a circle with radius r.

The area of a sector of a circle is found by $A = \frac{r^2\theta}{2}$, where r is the radius and θ is the angle that sweeps out the sector. Applying this to the ellipse, we obtain:

$$dA = \frac{r^2 d\theta}{2}$$

Dividing by dt, the equation becomes:

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt}$$

Substituting in equation (10), we now have:

$$\frac{dA}{dt} = \frac{r^2}{2}\left(\frac{h}{r^2}\right) \qquad \text{or} \qquad \frac{dA}{dt} = \frac{h}{2} \tag{16}$$

Therefore, $\frac{dA}{dt}$ is a constant. This proves Kepler's Second Law, which states that over equal periods of time, a planet sweeps out equal areas.

Kepler's Third Law

Beginning with equation (16) and integrating both sides, we have:

$$A = \frac{h}{2}t + k$$

When $t = 0$, the area A swept out by the planet must also be 0, so $k = 0$, and the equation becomes:

$$A = \frac{h}{2}t$$

Now let $t = T$, where T is the period of the planet, i.e. the amount of time required for the planet to complete one revolution around the sun. At time T, the planet has swept out the entire area of the ellipse, found by $ab\pi$. Thus:

$$\frac{h}{2}T = A = ab\pi$$

Substituting $b = a\sqrt{1-e^2}$ into the equation, and squaring both sides, we have:

$$\frac{h^2}{4}T^2 = a^3\left(a\left(1-e^2\right)\right)\pi^2 \tag{17}$$

We recall h^2 in equation (12), and rearranging, we have:

$$h^2 = \frac{eGM}{B}$$

Substituting this into equation (17) yields:

$$e\frac{GMT^2}{4B} = a^3\left(a\left(1-e^2\right)\right)\pi^2$$

Substituting this into the left side of equation (17) yields:

$$e\frac{GMT^2}{4}\frac{a\left(1-e^2\right)}{e} = a^3\left(a\left(1-e^2\right)\right)\pi^2$$

This easily simplifies to:

$$T^2 = \frac{4\pi^2}{GM}a^3$$

Thus, Kepler's Third Law states that the square of the period is proportional to the cube of the semi-major axis of the elliptical orbit.

Showing dM/dt is constant

Let

$$M = E - e\sin E$$

To prove that M varies uniformly with respect to time, it is necessary to show that $\frac{dM}{dt}$ is a constant. Differentiating gives us:

$$\frac{dM}{dt} = \frac{dE}{dt}(1 - e\cos E) \tag{18}$$

Using the chain rule and equations (2), (3) and (9) we find:

$$\frac{dE}{dt} = \frac{h\sin\theta}{a^2(1-e^2)\sin E}$$

Substituting this in equation (18) yields:

$$\frac{dM}{dt} = \frac{h\sin\theta(1 - e\cos E)}{a^2(1-e^2)\sin E} \tag{19}$$

Now we eliminate the variable E temporarily. To do this, we find the ratio of y_e to y_c using the basic equations for ellipses and circles in basic Cartesian coordinates:

$$\frac{x^2}{a^2} + \frac{{y_e}^2}{b^2} = 1$$
$$x^2 + {y_c}^2 = a^2$$

Solving for y_e and y_c, we find $\frac{y_e}{y_c}$ to be:

$$\frac{y_e}{y_c} = \frac{b}{a}$$

The ratio can also be expressed in polar coordinates as:

$$\frac{y_e}{y_c} = \frac{r\sin\theta}{a\sin E}$$

Equating these two, we have:

$$\sin\theta = \frac{b\sin E}{r}$$

Now we substitute this into equation (19) to produce:

$$\frac{dM}{dt} = \frac{h(1-e\cos E)}{rb}$$

Now we use equation (2) to eliminate the variable r and equation (3) to eliminate the variable E. The equation now becomes:

$$\frac{dM}{dt} = \frac{h}{ab}$$

By substituting h to solve in terms of T we have:

$$\frac{dM}{dt} = \frac{2\pi}{T}$$

This shows that the derivative of M with respect to time is a constant. It can be integrated to read:

$$M = \frac{2\pi t}{T}$$

SPHERICAL TRIGONOMETRY

<u>Introduction</u>

The universe can be viewed as a celestial sphere rotating around a stationary Earth. All heavenly bodies are represented by points on this sphere. Everyday, the celestial sphere rotates once around the Earth, changing the position of the heavenly bodies in the sky. By finding the relationship between points on the celestial sphere, one can plot the positions of such celestial bodies. To accomplish this, it is necessary to find the equivalents of the Law of Cosines and the Law of Sines for spherical triangles.

<u>Terminology</u>

A great circle is a circle that shares the same radius as the sphere it lies on. The shortest distance between two points on the surface of a sphere can be found by taking the arc of a great circle passing through both points. A spherical triangle is composed of three points on a sphere connected by arcs of great circles. The angle between two arcs is defined as the angle between the tangents to their common point.

<u>Proof for the Law of Cosines for Spherical Triangles</u>

It is possible to derive the Law of Cosines for Spherical Triangles from Fig. 4.

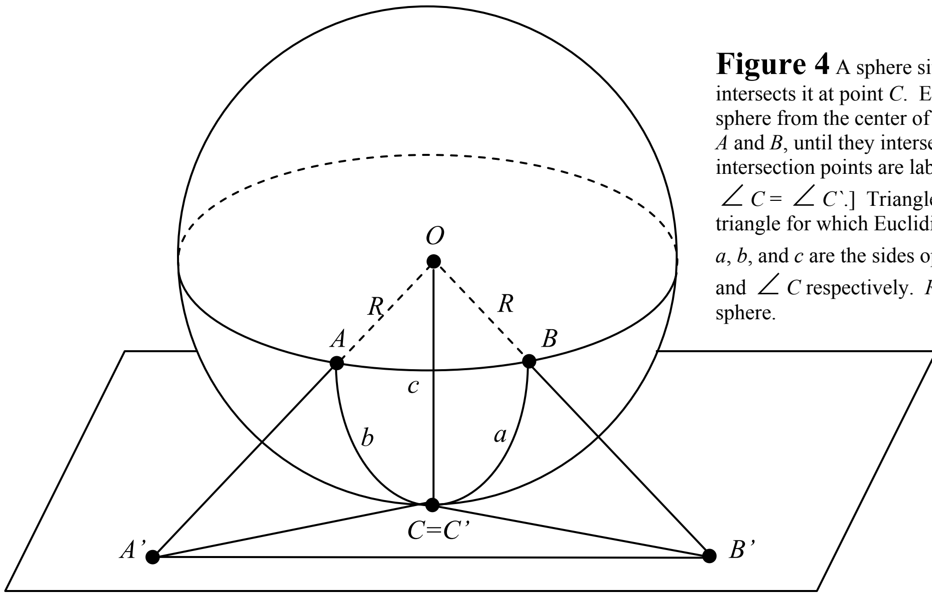

Figure 4 A sphere sits on a plane and intersects it at point *C*. Extend the radii of the sphere from the center of the sphere to the points *A* and *B*, until they intersect the plane. These intersection points are labeled *A*` and *B*`. [Note: $\angle C = \angle C`$.] Triangle *A`B`C`* is a planar triangle for which Euclidian geometry applies. *a*, *b*, and *c* are the sides opposite $\angle A$, $\angle B$, and $\angle C$ respectively. *R* is the radius of the sphere.

Knowing that $\frac{S}{R} = \theta$ where S is the length of an arc of a great circle and θ is the central angle of that arc in radians, $\angle BOC = \frac{a}{R}$, $\angle AOC = \frac{b}{R}$, and $\angle BOA = \frac{c}{R}$.

Finding the Lengths of OB`, OA`, B`C`, and A`C`

Figure 5

Using basic trigonometric definitions for Euclidean Triangles on $\Delta OB`C`$ and $\Delta OA`C`$:

$$\tan\frac{a}{R} = \frac{C'B'}{R} \qquad \sec\frac{a}{R} = \frac{OB'}{R}$$

$$C'B' = R\tan\frac{a}{R} \qquad OB' = R\sec\frac{a}{R}$$

$$\tan\frac{b}{R} = \frac{C'A'}{R} \qquad \sec\frac{b}{R} = \frac{OA'}{R}$$

$$C'A' = R\tan\frac{b}{R} \qquad OA' = R\sec\frac{b}{R}$$

Finding the Length of $A`B`$

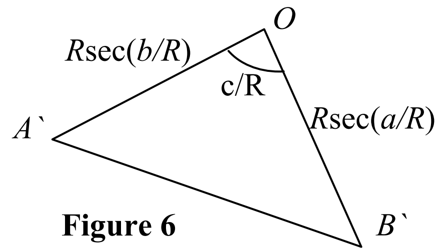

Figure 6

Note that $\angle A`OB` = \angle AOB = \frac{c}{R}$.

In $\Delta OA`B`$, solve for $A`B`$ using the Law of Cosines of Euclidean Triangles.

$$[A`B`]^2 = \left[R\sec\frac{b}{R}\right]^2 + \left[R\sec\frac{a}{R}\right]^2 - 2R^2\cos\frac{c}{R}\sec\frac{a}{R}\sec\frac{b}{R} \tag{20}$$

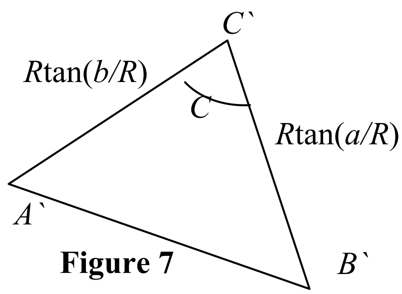

Figure 7

Similarly, in $\Delta A`B`C`$, Solve for $A`B`$.

$$[A`B`]^2 = \left[R\tan\frac{b}{R}\right]^2 + \left[R\tan\frac{a}{R}\right]^2 - 2R^2\tan\frac{a}{R}\tan\frac{b}{R}\cos C\,. \tag{21}$$

Obtaining the Spherical Law of Cosines

$\Delta A`B`C$ and $\Delta OA`B`$ have one side in common, $A`B`$. Set equations (20) and (21) equal to each other.

$$\left[R\sec\frac{b}{R}\right]^2 + \left[R\sec\frac{a}{R}\right]^2 - 2R^2\cos\frac{c}{R}\sec\frac{a}{R}\sec\frac{b}{R} = \left[R\tan\frac{b}{R}\right]^2 + \left[R\tan\frac{a}{R}\right]^2 - 2R^2\tan\frac{a}{R}\tan\frac{b}{R}\cos C$$

Substitute $1+\tan^2 x = \sec^2 x$. Simplify and solve for $\cos\frac{c}{R}$.

As a result, the Law of Cosines of Spherical Triangles is:

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cos\frac{b}{R} + \sin\frac{a}{R}\sin\frac{b}{R}\cos \mathrm{C}. \tag{22}$$

where R is the radius of the sphere and a/R, b/R, and c/R are the internal central angles.

Proof for the Law of Sines for Spherical Triangles

From the Law of Cosines for spherical triangles, it is possible to derive the Law of Sines for spherical triangles through algebraic manipulations.

Solve equation (22) for cos C.

$$\frac{\cos\frac{c}{R} - \cos\frac{a}{R}\cos\frac{b}{R}}{\sin\frac{a}{R}\sin\frac{b}{R}} = \cos C$$

Square both sides of the equation and substitute $\cos^2 C = 1 - \sin^2 C$.

$$\left[\frac{\cos\frac{c}{R} - \cos\frac{a}{R}\cos\frac{b}{R}}{\sin\frac{a}{R}\sin\frac{b}{R}}\right]^2 = 1 - \sin^2 C$$

Manipulate this equation into a form similar to the Laws of Sines for Euclidean Triangles.

Isolate $\sin^2 C$ and divide both sides of the equation by $\sin^2\frac{c}{R}$.

$$\frac{\sin^2\frac{a}{R}\sin^2\frac{b}{R} - \cos^2\frac{c}{R} + 2\cos\frac{a}{R}\cos\frac{b}{R}\cos\frac{c}{R} - \cos^2\frac{a}{R}\cos^2\frac{b}{R}}{\sin^2\frac{a}{R}\sin^2\frac{b}{R}\sin^2\frac{c}{R}} = \frac{\sin^2 C}{\sin^2\frac{c}{R}}$$

Substitute all of the $\cos^2 x$ terms with $1 - \sin^2 x$ and simplify.

$$\frac{\sin^2 C}{\sin^2\frac{c}{R}} = \frac{-2 + \sin^2\frac{a}{R} + \sin^2\frac{b}{R} + \sin^2\frac{c}{R} + 2\cos\frac{a}{R}\cos\frac{b}{R}\cos\frac{c}{R}}{\sin^2\frac{a}{R}\sin^2\frac{b}{R}\sin^2\frac{c}{R}}$$

a, b, c are interchangeable, as are A, B, C, with the right side of the equation remaining constant.

Thus, the Law of Sines for Spherical Triangles is:

$$\frac{\sin C}{\sin \frac{c}{R}} = \frac{\sin B}{\sin \frac{b}{R}} = \frac{\sin A}{\sin \frac{a}{R}}. \tag{23}$$

Positioning Points on a Sphere

We can view the heavenly bodies as points on the sphere rotating about the Earth, which remains stationary at the center of this sphere. We then extend the North and South Poles to the celestial sphere to create the North and South Celestial Poles. We also extend the Earth's equator to create the celestial equator. We create an arc connecting the North and South Celestial Poles passing through the vernal equinox (the sun's position on the first day of spring) to create a reference point. Using the celestial equator and this arc, we develop a system for locating any point on the celestial sphere. We imitate the Earth's latitude and longitude system. The declination (δ) corresponds to latitude and is measured in degrees (-90° to +90°); The right ascension (α) corresponds to longitude and is measured in hours (24 hours = 360°). As one hour on Earth passes, the celestial sphere effectively rotates 15° around the Earth.

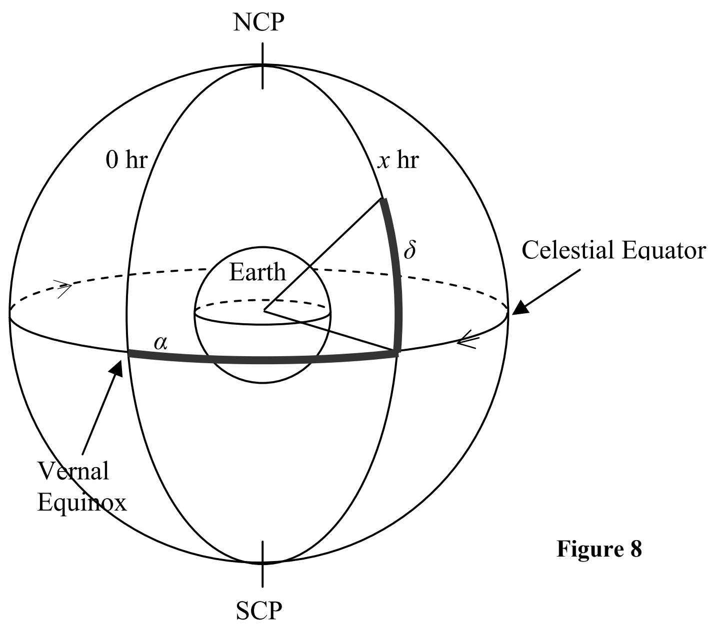

Figure 8

Determining the Declination and Right Ascension of the Sun

The sun viewed from the Earth appears to take a path along the celestial sphere at an angle to the celestial equator. This path is termed the ecliptic. The declination of the sun at the point farthest from the celestial equator is a physical constant ε = 23° 27`. The points where the ecliptic and celestial equator intersect occur at the beginning of spring and fall. The arc passing through the vernal equinox is marked as the 0hr and is used as the reference arc used for right ascension measurements. We draw a perpendicular to the ecliptic passing through the Earth to create the North and South Ecliptic poles.

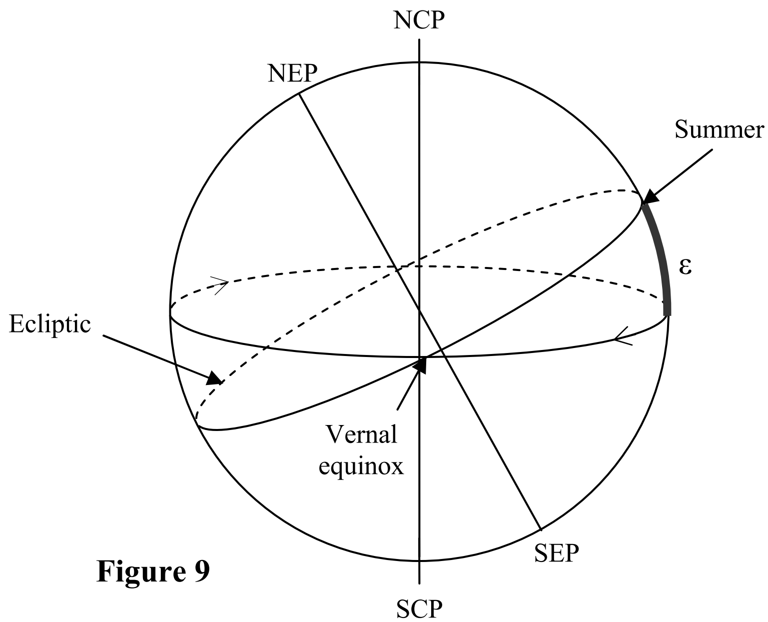

Figure 9

Next, we want to be able to give the declination and right ascension of the sun on the ecliptic at any time, given the angle λ (Fig. 10). We use the previously derived Spherical Law of Sines to achieve this goal.

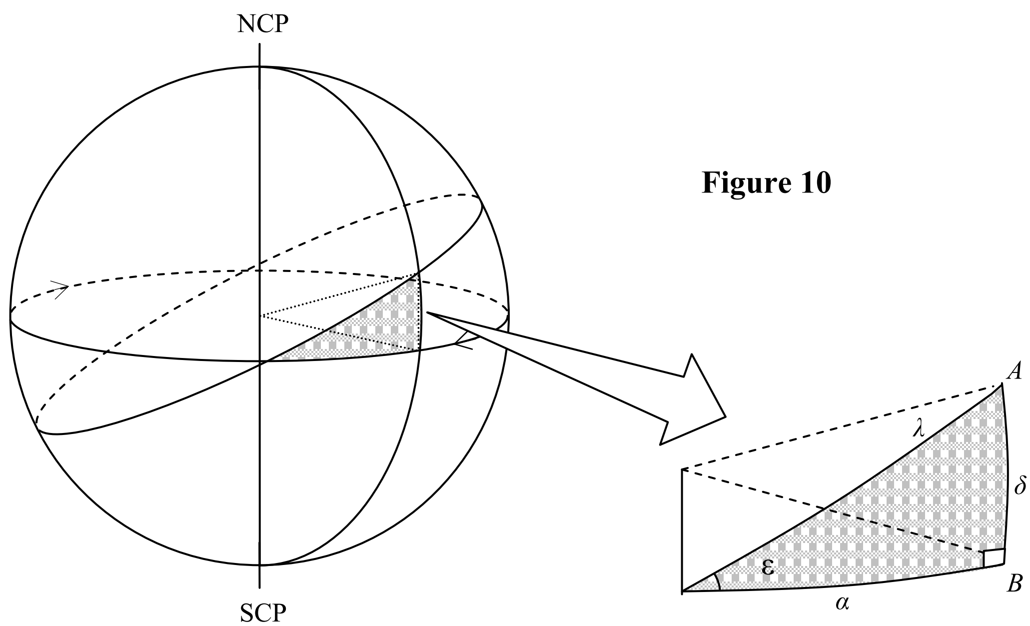

Figure 10

From the Spherical Law of Sines:

$$s \;=\; r\,\Theta \qquad\qquad \ell\delta \;=\; r\,\delta \qquad\qquad \frac{\ell\delta}{R} = \delta$$

$$\frac{\cos\varepsilon}{\sin\lambda} = \frac{\sin 90^\circ}{\sin\lambda} = \frac{\sin\varepsilon}{\sin\delta}$$

$$\sin\delta = \sin\varepsilon\sin\lambda \qquad\qquad \sin\alpha = \cos\varepsilon\sin\lambda$$

Relating Equatorial and Ecliptic Coordinates

Next, we develop a coordinate system using the ecliptic, North Ecliptic Pole, and South Ecliptic Pole. The ecliptic latitude is given by β, and the ecliptic longitude is given by λ. We then develop equations giving equatorial coordinates from the ecliptic coordinates by using spherical trigonometry.

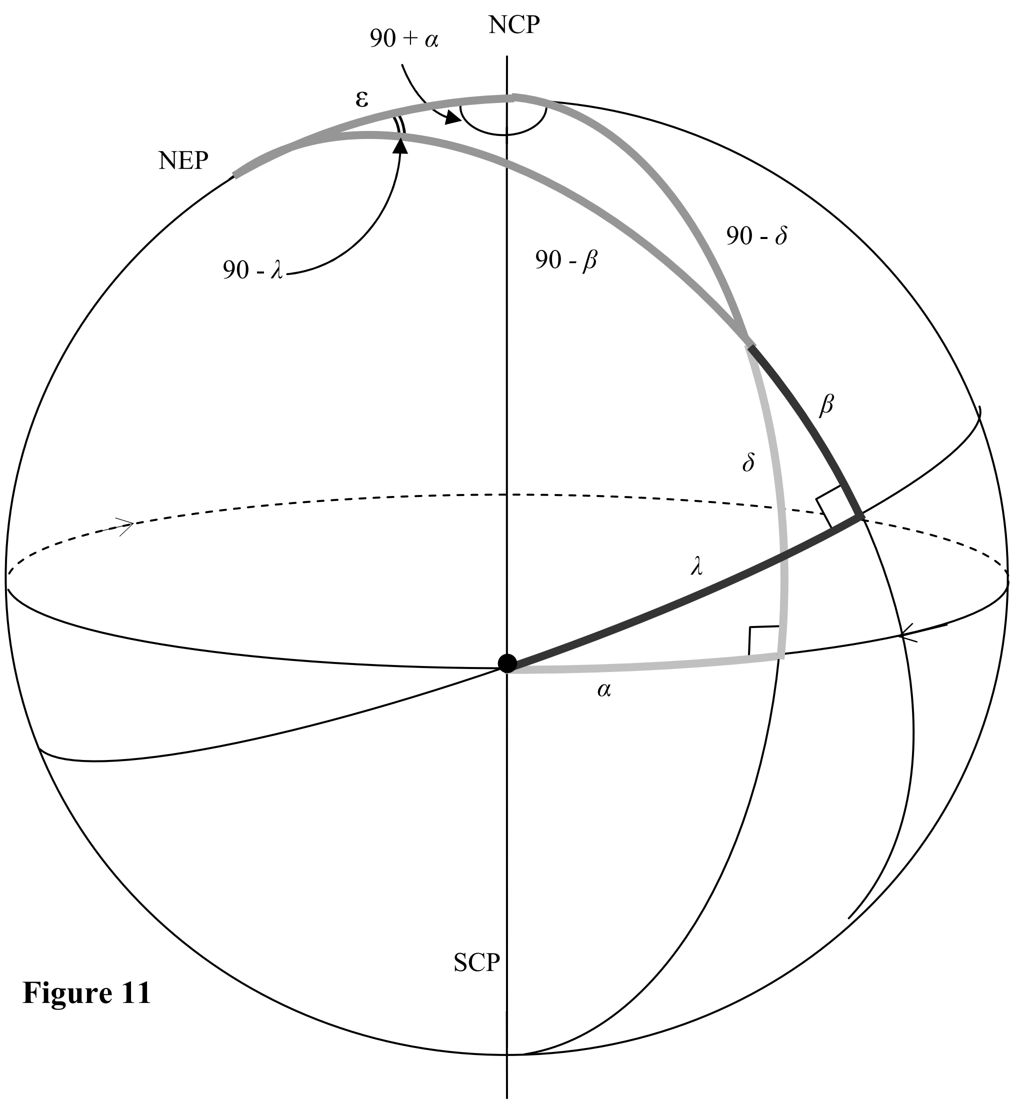

Figure 11

We first use the Law of Cosines to find δ in terms of β, λ, and ε.

$$\cos(90° - \delta) = \cos(90° - \beta)\cos\varepsilon + \sin\varepsilon\sin(90° - \beta)\cos(90° - \lambda)$$
$$\sin\delta = \sin\beta\cos\varepsilon + \sin\varepsilon\cos\beta\sin\lambda$$

Now that we have δ, we can use the Law of Sines to find α in terms of β, δ, and ε.

$$\frac{\sin(90+\alpha)}{\sin(90-\beta)} = \frac{\sin\varepsilon}{\sin(90-\delta)}$$
$$\sin(90+\alpha) = \frac{\sin\varepsilon\cos\beta}{\cos\delta}$$

Finding the Sunrise and Sunset

Next we relate the position of the sun at sunrise/sunset at any latitudinal position on the Earth. To do this we extend the horizon at that point to the celestial sphere. The hour angle *H* (Fig. 12) gives the time from noon until sunrise/sunset. We can find *H* for a certain latitudinal position given the declination of the sun on any day, and thus find the sunrise/sunset on that day.

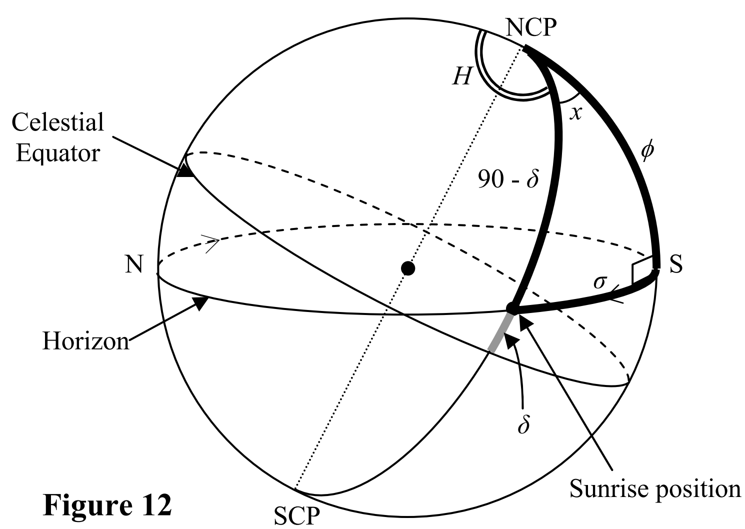

Figure 12

$$x = 180 - H \qquad \cos(90° - \delta) = \cos\phi\cos\sigma + \sin\phi\sin\sigma\cos(90°)$$
$$\cos x = -\cos H \qquad \cos\sigma = \frac{\sin\delta}{\cos\phi}$$

Now substitute:

$$\cos\sigma = \sin\delta\cos\phi + \cos\delta\sin\phi\cos x$$
$$\frac{\sin\delta}{\cos\phi} = \sin\delta\cos\phi + \cos\delta\sin\phi(-\cos H)$$
$$\cos H = -\tan\delta\tan\phi$$

Finding Theta

To find θ**,** the angle that a planet has traveled away from perihelion, we must start with a time *t*, measured in days since perihelion. With this value of *t*, we can substitute into the equation:

$$M = \frac{2\pi t}{T}$$

where *T* is the period of the planet. With all of this, we can find the value of *M*, which is defined to be:

$$M = E - e \cdot \sin E$$

Rearranging this equation, we can find an iterative process to approximate the value of *E*.

$$E_{n+1} = M + e \cdot \sin E_n$$

By allowing E_0 to equal *M*, we can quickly find an accurate value for *E*. This value of *E* allows us to solve for θ, because of the relationship between *E* and θ:

$$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)$$

Solving this gives us a value for θ.

Finally, we want to relate the position of the sun from perihelion (θ) to the position of the sun from the vernal equinox (λ) using the known value ω. Figure 13 illustrates visually their relationship.

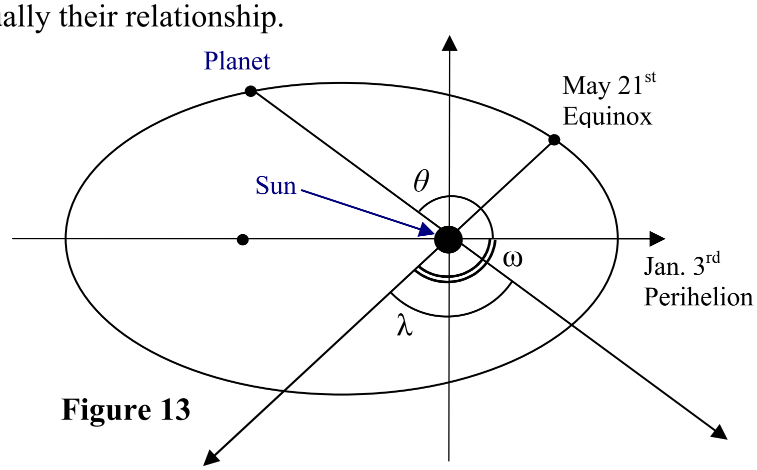

Figure 13

Using three equations, we can predict the sunrise and sunset on any given day. We make a calculator program to find these times values of any day, and accurately predict the sunrise and sunset of August 2nd, 2003 to be 5:55am and 8:09pm.

$\lambda = \theta + w - 180°$	$w = 101°$
$\sin\delta = \sin\varepsilon \sin\lambda$	$\varepsilon = 23°27`$
$\cos H = -\tan\delta \tan\phi$	$\phi = 40°46`$

CONCLUSION

By utilizing geometric and mathematical relationships, we have been able to develop a series of equations that govern the movement of heavenly bodies. Both geocentric and heliocentric models have proved valuable in our examination of celestial mechanics. Using the relationships that we have derived, one can predict the events such as sunrise and sunset and plot the location of a planet at a given point in its revolution. Our examination of Kepler's Laws and spherical triangles reveals that planetary motion is governed by definite mathematical principles and can be explained using systems of equations.

During this month, we learned about the process of how one would make new connections and derive new laws. We learned that segmenting a complicated problem makes it easier to solve (Oh wait, we learned that in Kindergarten). Also, sharing is important (Work… of course), and during the learning process, being negative is a bad, bad, BAD thing. We are all better people now that we have completed this project (Aww, I feel all warm and fuzzy inside now!).