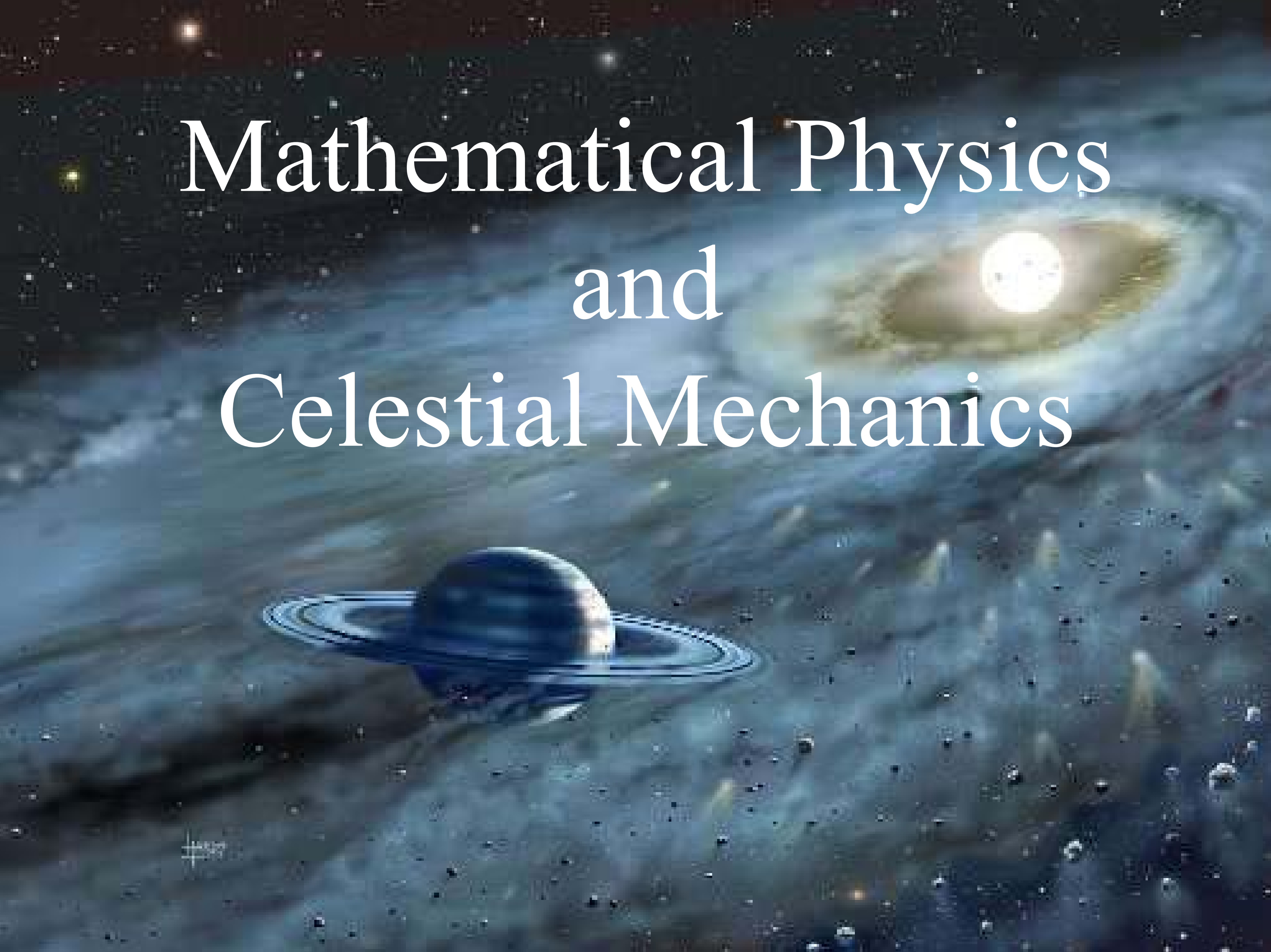
# Mathematical Physics and Celestial Mechanics



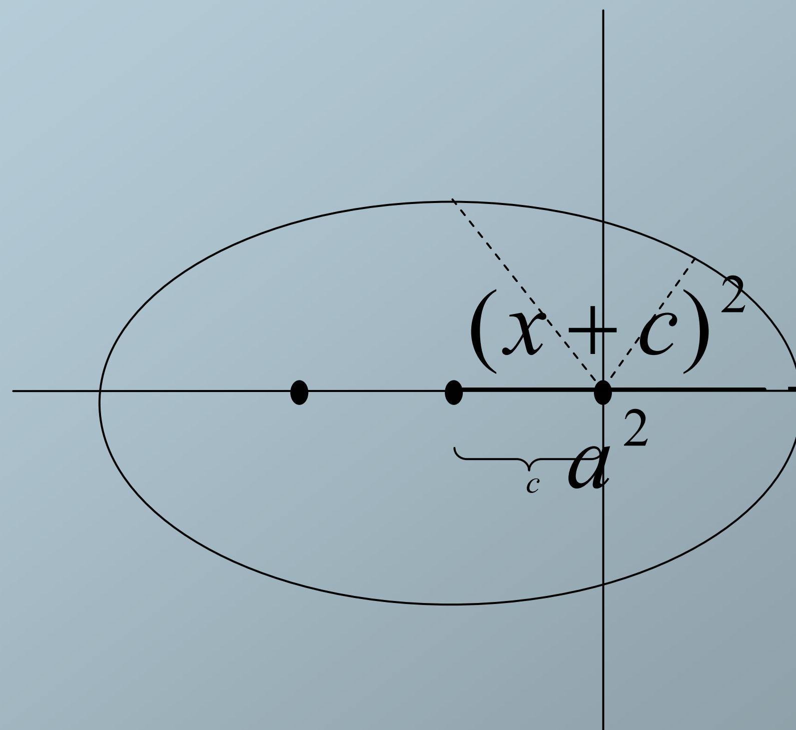
# Goals

- To trace out the orbits of the planets around the sun
- To predict sunrise and sunset

# Outline

- Elliptical Geometry
- Kepler's Laws
- Spherical Trigonometry
- Applications

# Elliptical Geometry



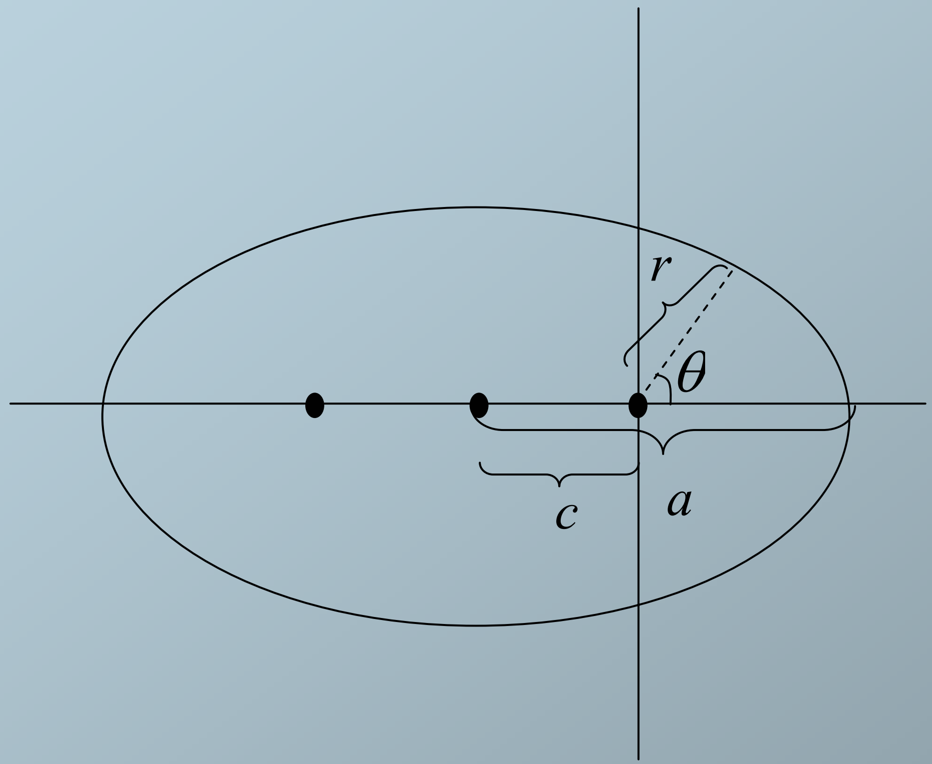

$$\frac{(x+c)^2}{a^2}+\frac{y^2}{b^2}=1$$

Graph of an elliptical orbit, shifted so that the sun is a focus at the origin

Equation of this ellipse in Cartesian coordinates

# Elliptical Geometry

Equation relating the distance of the planet from the origin to angle θ




$$eccentricity = e \equiv \frac{c}{a}$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

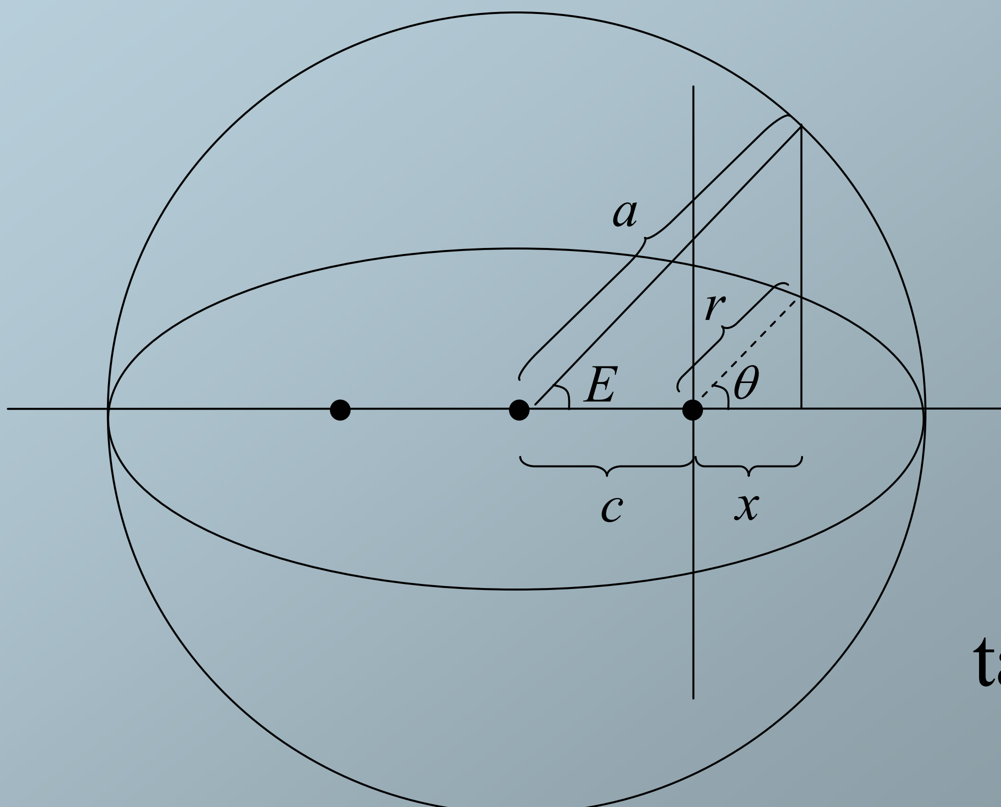
Polar coordinates

# Elliptical Geometry

A circle is positioned about the ellipse.

The planet’s location is projected onto the circle.

From here, one can derive the relationship between angles E and θ.




$$\cos E = \frac{\ldots s\,\theta + e}{\ldots e\cos\theta}$$

$$\tan\frac{E}{2} = \tan\frac{\theta}{2}\sqrt{\frac{1-e}{1+e}}$$

# Kepler's Laws of Planetary Motion

- Planets travel in elliptical orbits.
- Equal areas are swept out over equal times.
- The square of the period is directly proportional to the cube of the semi-major axis.

# Kepler's First Law

- The force on a planet is caused by gravity.
- Force is separated into components.
- These vectors are converted to polar coordinates.
- Manipulation yields …

$$\mathrm{F} = -\frac{GMm}{\mathrm{r}^2}$$

M = Mass of the sun

m = Mass of the planet

G = Gravitational constant

h = a constant of integration

$$\mathrm{F}_\mathrm{x} = -\frac{GMm}{\mathrm{r}^2}\cos\theta = m\frac{d^2}{dt^2}(r\cos\theta)$$

$$\mathrm{F}_\mathrm{y} = -\frac{GMm}{\mathrm{r}^2}\sin\theta = m\frac{d^2}{dt^2}(r\sin\theta)$$

$$\sin\theta\frac{d^2r}{dt^2}+2\cos\theta\frac{dr}{dt}\frac{d\theta}{dt}+r\cos\theta\frac{d^2\theta}{dt^2}-r\sin\theta\left(\frac{d\theta}{dt}\right)^2=-\frac{GM}{r^2}\sin\theta$$

$$2\frac{dr}{dt}\frac{d\theta}{dt}-r\frac{d^2\theta}{dt^2}=0$$

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-\frac{GM}{r^2}$$

$$\frac{2}{r}\frac{dr}{dt}=-\frac{1}{p}\frac{dp}{dt}$$

$$2\ln r=-\ln p+c$$

$$r^2=\frac{e^c}{p} \qquad pr^2=h$$

$$r^2\frac{d\theta}{dt}=h \qquad \frac{d\theta}{dt}=\frac{h}{r^2}$$

$$\frac{d^2r}{dt^2}-\frac{h^2}{r^3}=-\frac{GM}{r^2}$$

$$\frac{dr}{dt}=\frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt}=-h\frac{du}{d\theta}$$

$$\frac{d^2r}{dt^2}=-h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{h}{r^2}=h^2u^2\frac{d^2u}{d\theta^2}$$

$$\frac{d^2u}{d\theta^2}+u=\frac{GM}{h^2}$$

$$u=B\cos(\theta+C)+\frac{GM}{h^2}$$

$$r=\frac{\frac{h^2}{GM}}{1+\frac{Bh^2}{GM}\cos(\theta+C)}$$

$$r=\frac{\frac{h^2}{GM}}{1+\frac{Bh^2}{GM}\cos\theta}$$

# Kepler's First Law

- Eccentricity is also

$$e = \frac{Bh^2}{GM}$$

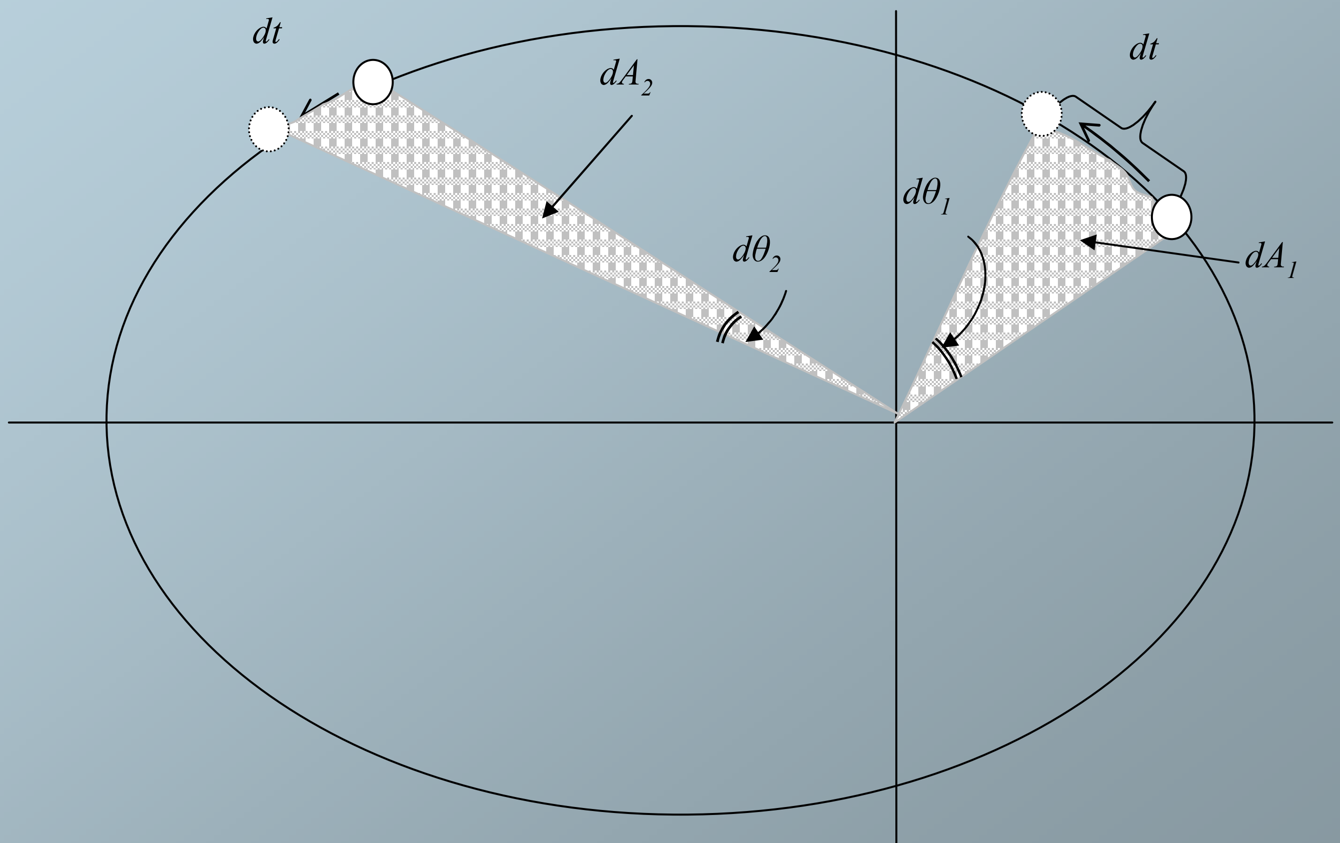
- When $e$ is applied to the current equation, it yields →

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

- This formula is that of an ellipse, thus proving Kepler's first law.

# Kepler's Second Law

As a planet moves, it sweeps out an area inside its orbit. Over equal times, this area is constant.




$$dA_1 = dA_2$$

$$\frac{dA}{dt} = \frac{h}{2}$$

= a constant

# Kepler’s Third Law

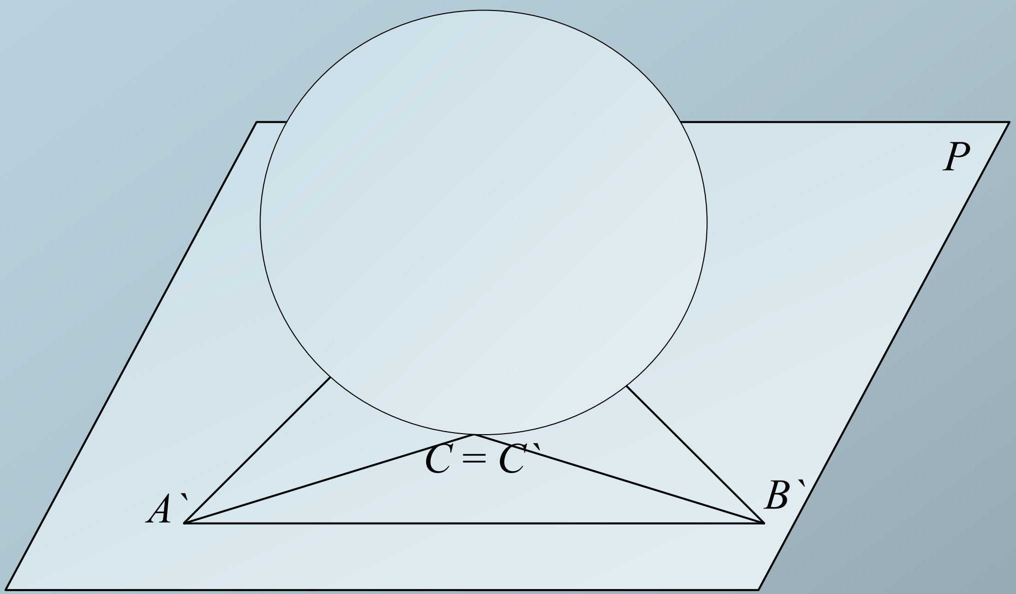
- The square of the period is directly proportional to the cube of the semi-major axis.
- From the final formula of Kepler’s Second Law, we can prove the third law.

$$A = \frac{h}{2} t \qquad \longrightarrow \quad e \frac{GMT^2}{4} \quad \frac{h^2}{4} \quad \frac{a(1-e^2)}{e} \quad T^2 = \frac{4\pi^2}{GM} a^3 \quad a^3\left(1-e^2\right)\pi^2$$

# Spherical Trigonometry

Prove:

## Law of Cosines




Using the diagram and the Law of Sines of Euclidean Triangles, find the sides and angles of two triangles that share a side.

Solve for the common side for both triangles and set them equal to each other.

Through algebraic manipulation, solve for cos (c/R).

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cos\frac{b}{R} + \sin\frac{a}{R}\sin\frac{b}{R}\cos C$$

$$\left[R\sec\frac{b}{R}\right]^2 + \left[R\sec\frac{a}{R}\right]^2 - 2R^2\cos\frac{c}{R}\sec\frac{a}{R}\sec\frac{b}{R} = \left[R\tan\frac{b}{R}\right]^2 + \left[R\tan\frac{a}{R}\right]^2 - 2R^2\tan\frac{a}{R}\tan\frac{b}{R}\cos C$$

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# Finally!

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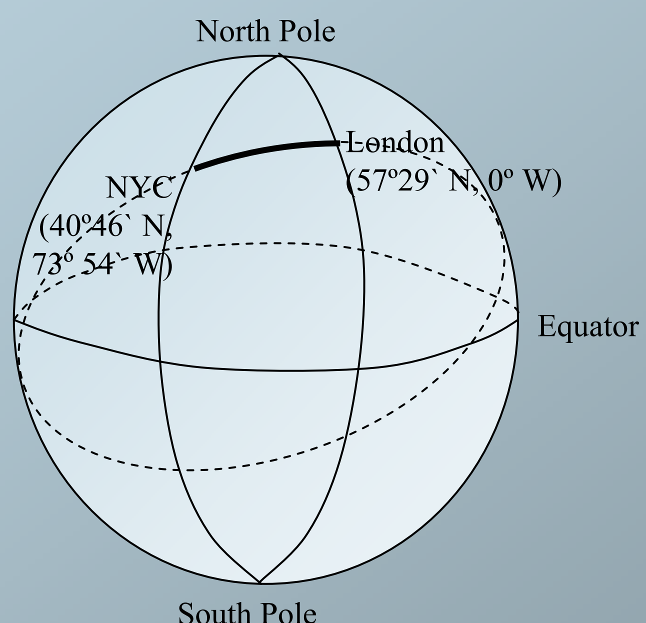
# Spherical Trigonometry

## Application: Find the Shortest Distance Between 2 Cities

The shortest distance is on an arc of a great circle:



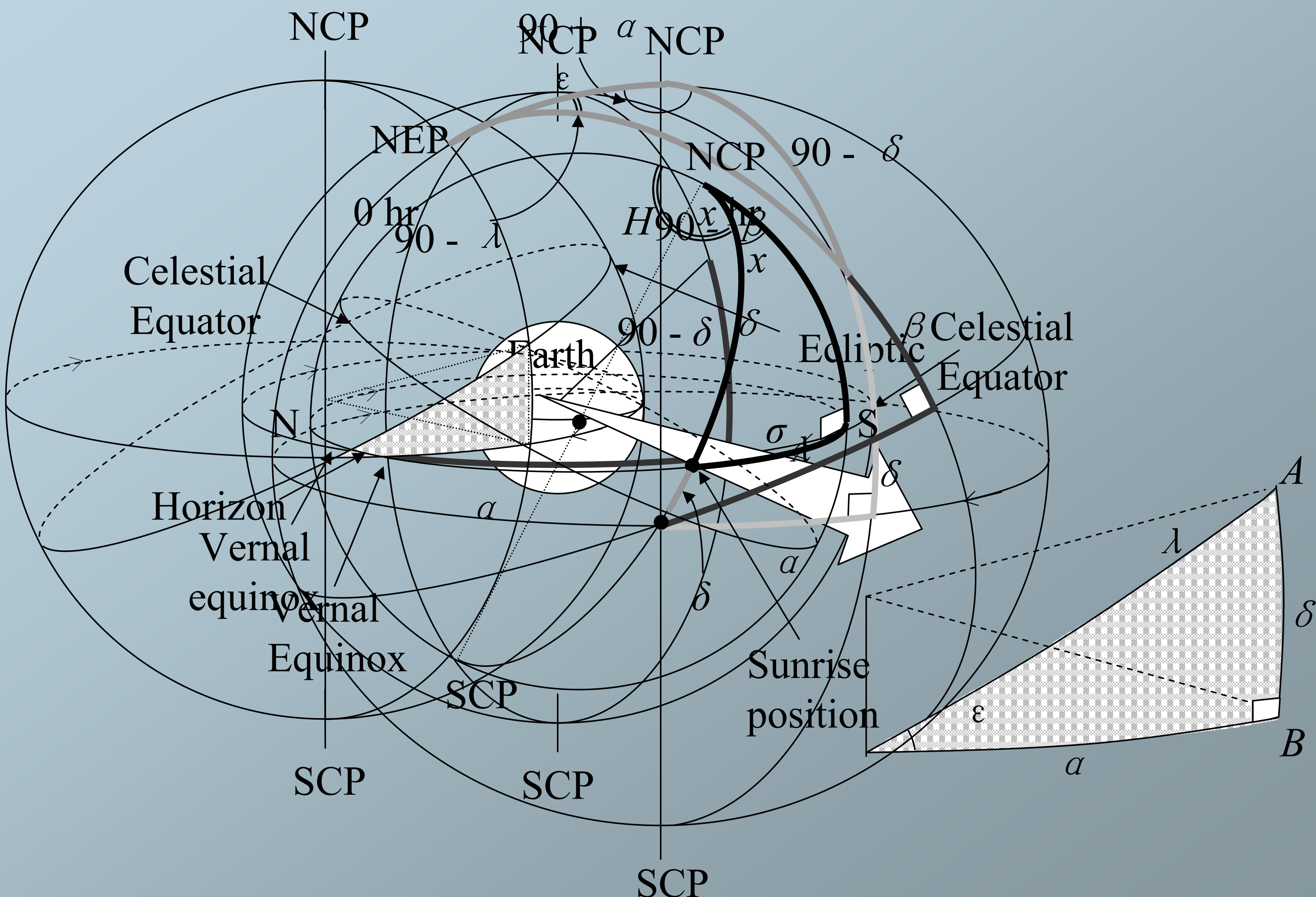

Given are the longitude and latitude of the two cities.

The Law of Cosines can be used on two spherical triangles with a shared angle to find the distance between two points.

3460 Miles Derived

3471 Miles Actual

# Spherical Trigonometry



# Spherical Trigonometry

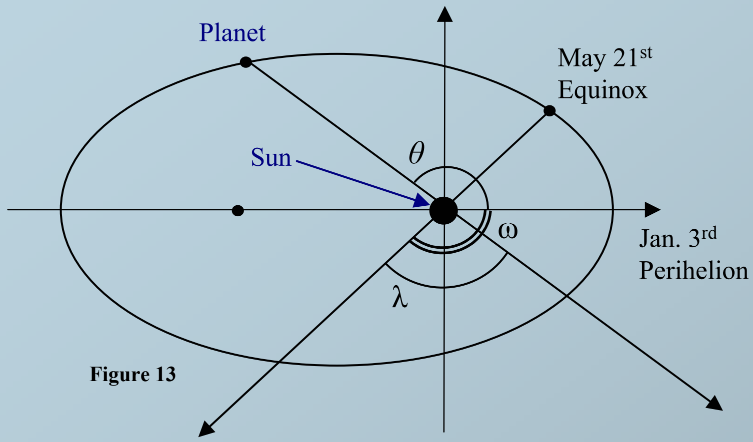


**Figure 13**

## Sunrise/Sunset

Derivation of formulas come from applying the Law of Sines and the Law of Cosines to the previously shown figures.

The times for sunrise and sunset can be found using these three equations.

$$\lambda = \theta + \omega - 180^\circ$$

$$\sin\delta = \sin\varepsilon \sin\lambda$$

$$\cos H = -\tan\delta \tan\phi$$

$$\omega_{Earth} = 101^\circ$$

$$\varepsilon_{Earth} = 23^\circ 27`$$

$$\phi_{Madison} = 40^\circ 46`$$

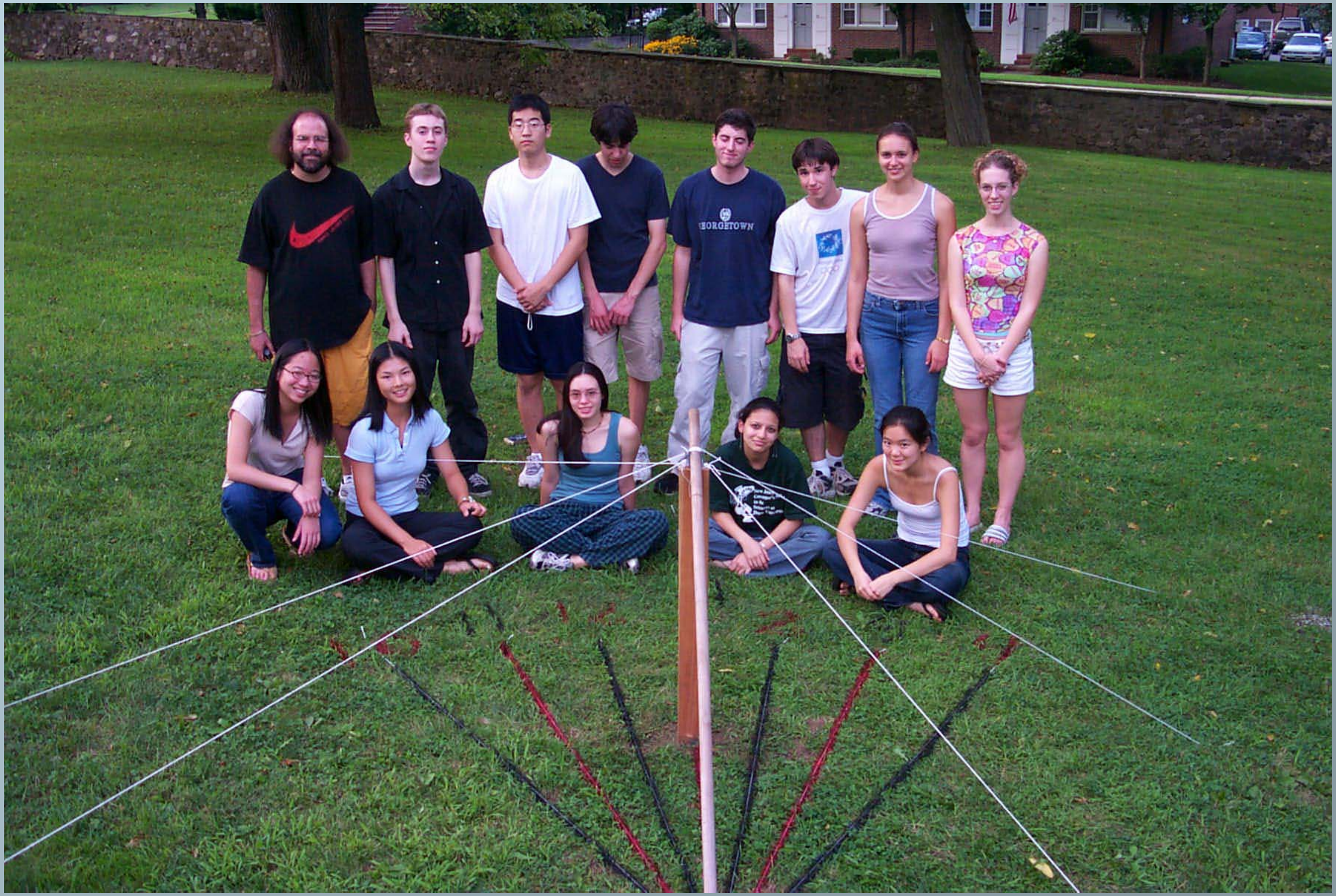
α = right ascension

δ = declination

# Conclusion

- We learned how to apply mathematical laws to physics
- Planetary orbits can be described using mathematical formulae
- You can’t make a sundial without sun
- Drew rarely has sun
- It is not good for sundial construction

# The rain went away!



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