CELESTIAL MECHANICS

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ABSTRACT

This team project was devoted to modeling the orbits of heavenly bodies. Our objectives were divided into three components, but the ultimate goal was to create a series of equations that could be used to determine the position in space of any planet at any date and time, relative to the sun. The entire group was divided into three smaller groups, each with a particular specialization, who worked independently at first but eventually combined their data and supported each other's findings. One group worked with ellipse geometry, another proved Kepler's Laws, and the third explored spherical trigonometry. The group discovered different equations to model these elements of celestial mechanics and used them to find the planetary positions at a particular date and time. The individual groups produced the same answer.

INTRODUCTION

Despite the seeming simplicity of stargazing, the study of Celestial Mechanics assimilates several areas of mathematics into a vast and complex whole. The primary goal of these calculations lies in locating and understanding the position of celestial bodies as seen from Earth. Through a combination of spherical trigonometry, ellipse geometry, and calculus, equations can be determined that allow us to ascertain the positions of the sun and planets in relation to one another at any given date and time.

We began this project with an understanding of trigonometry as it relates to flat triangles, but starting with a basis only in two-dimensional figures, the laws of spherical trigonometry have to be deduced. In trigonometry, every high school student learns how to determine all the angles and sides of a planar triangle when given certain limited pieces of information. On a flat plane, it is relatively simple, using the law of sines and the law of cosines, to calculate the sides and angles not given. But when triangles are drawn on a sphere, new laws must be discovered to calculate the resulting values. If this sphere is the celestial sphere (the view of space as seen from Earth), it becomes possible with these new laws to plot the location of any object in space. It is not sufficient, though, to describe stationary objects in space; celestial bodies are constantly in motion. Ellipse geometry is necessary to describe mathematically the path of the Earth as it orbits the sun. Combined with Kepler's laws of planetary motion, derived using calculus, we can find the angles between a north-south reference line (on the surface of the Earth) and the sun at a specified date and time. The practical application of such calculations lies in the construction of sundials, a practice that has been used for timekeeping since biblical times.

The following project addresses the problem of combining and simplifying this mass of mathematics into a few equations that can be used to achieve practical ends. First of all, they can be incorporated into a computer program which can calculate r and ϑ, two values that describe the position of the sun in relation to Earth. Furthermore, they can be used to translate the placement of a sundial's shadow into the angle of the sun, and from there, the time. This practical application of celestial mechanics is only possible after extensive calculations, and these calculations are the focus of the following article.

ELLIPSE GEOMETRY

Finding the Value of r as a Function of ϑ.

Our initial goal was to write an equation for the Earth's elliptical orbit, considering that the standard equation for an ellipse centered at the origin is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where *a* is the semi-major and *b* the semi-minor axis, and the value of *a* is always greater than that of *b*. We can move this equation so that the origin corresponds to a focus rather than the center of the ellipse, an important change because it will later be proven that the sun lies at a focus of the ellipse. When rearranged, the equation becomes

$$\frac{(x+c)^2}{a^2}+\frac{y^2}{b^2}=1$$

where *c* is the distance from the center to either focus. Changing these rectangular coordinates to polar coordinates, the equation is

$$\frac{(r\cos\vartheta+c)^2}{a^2}+\frac{r^2\sin^2\vartheta}{b^2}=1$$

Rearranging the terms of this equation and substituting in using the known relationship $a^2-b^2=c^2$, we can simplify the equation to the following quadratic:

$$r^2(b^2\cos^2\vartheta+a^2\sin^2\vartheta)+r(b^2c\cos\vartheta)-b^4=0$$

Using the quadratic formula, two possible values of *r* can be calculated:

$$r=\frac{-b^2c\cos\vartheta\pm ab^2}{b^2\cos^2\vartheta+a^2\sin^2\vartheta}$$

At this point, ae can be substituted for *c*, where *e* is the eccentricity of the Earth's orbit, and a^2-c^2 can be substituted for b^2. When simplified, the following equation is obtained:

$$r=\frac{(ae\cos\vartheta)(e+1)(e-1)\pm a(1+e)(1-e)}{(1+e\cos\vartheta)(1-e\cos\vartheta)}$$

Recall that *r* is a distance, and therefore cannot be negative. Knowing that *e* must be between the values of zero and one, the positive operation in the numerator is the only one which will ensure that all possible values of *r* will be positive. Thus, the value of *r* that uses subtraction can be eliminated. Knowing this, the equation can be further simplified to

$$r=\frac{a(1-e^2)}{1+e\cos\vartheta}\qquad\text{(Eq. 1)}$$

Circumscribing a Circle about the Ellipse.

Recall that r is the distance from the sun to the planet Earth. Since the orbit of the Earth is elliptical, r is constantly changing. However, it is possible to circumscribe a circle around the ellipse. This is an advantageous step to take because the radius of the circle is constant. In addition, the diameter of the circle is equal in length to twice the length of the semi-major axis of the ellipse. Thus, the radius of the circle is a.

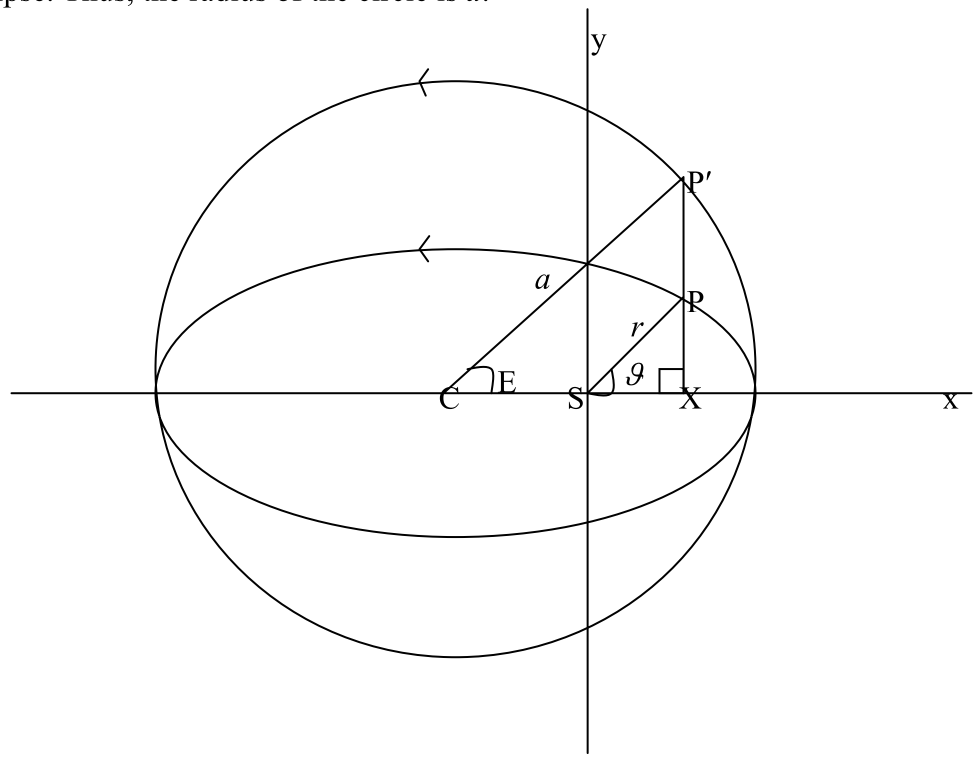

Figure 1 – Circle circumscribed about an ellipse

Figure 1 models this scenario, in which the elliptical orbit of the Earth around the sun is inscribed in a circle with radius a. The ellipse and the circle are both centered at C. The sun is located at S, the focus of the ellipse. The Earth is located on the ellipse at P, and P′ is located on the circle. A line can be drawn through P and P′ that is perpendicular to the x-axis. Using trigonometric functions, we find that the distance from C to X is $a\cos E$. This distance (C to X) is equal to the distance from the center of the ellipse to the focus (C to S) plus $r\cos\vartheta$ (S to X). Therefore, $a\cos E = r\cos\vartheta + c$, where c is the distance from the center to the focus. Bear in mind, c is related to the eccentricity (e) and the semi-major axis (a) through the equation $c = ae$. Using the equation $a\cos E = r\cos\vartheta + ae$ and the equation previously found for r where $r = \dfrac{a\left(1-e^2\right)}{1+e\cos\vartheta}$, we can solve for $\cos E$ and $\cos\vartheta$.

$$\cos E = \frac{\cos\vartheta + e}{1 + e\cos E}$$

$$\cos\vartheta = \frac{\cos E - e}{1 - e\cos E}$$

Using these two equations and the half-angle formula for tangent, $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$, it can be proven that $\tan\frac{E}{2} \propto \tan\frac{\vartheta}{2}$. After numerous algebraic calculations, it is found that:

$$\tan\frac{\vartheta}{2} = \sqrt{\frac{e+1}{1-e}}\tan\frac{E}{2}$$

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{e+1}}\tan\frac{\vartheta}{2} \qquad \text{(Eq. 2)}$$

These relations lack the dual sign possibility ($\pm$) of the half-angle formula because $\frac{\vartheta}{2}$ and $\frac{E}{2}$ are always in the same quadrant, since P and P′ are both located on a line perpendicular to the x-axis. Thus, the positive sign of the half-angle formula can be used in finding the relation between $\frac{\vartheta}{2}$ and $\frac{E}{2}$. These half-angle relations can be used to find ϑ if E is known and vice versa, and their form is convenient for future computer programming.

KEPLER'S LAWS OF PLANETARY MOTION

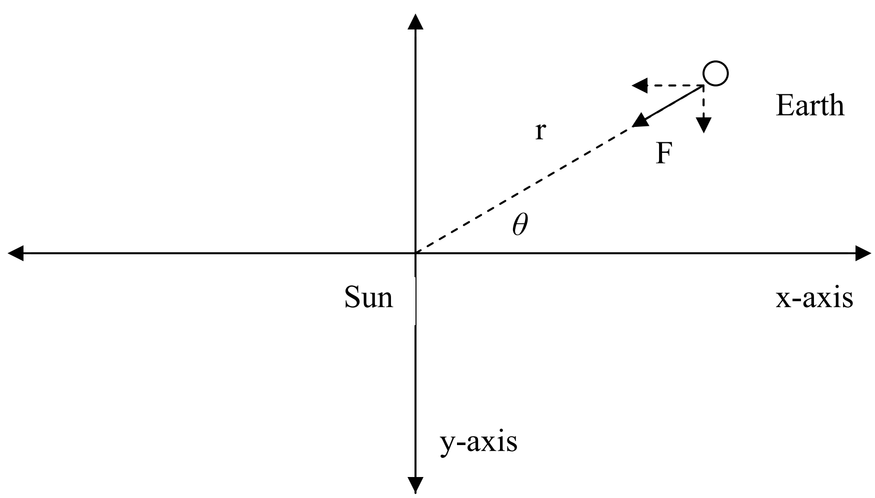

Figure 2 – Gravitational Force on the Earth

<u>Kepler's First Law</u>

Kepler's First Law states that the planets move in an ellipse with the sun at a focus. This is based on mathematics and Newton's Law of Gravity.

According to Newton's Law of Gravity, the gravitational force on the Earth is $F = \frac{GM_s m}{r^2}$, and from his second law of motion, we find that the force on the Earth is equal to its mass multiplied by its acceleration. In Figure 2, the Earth's movement around the sun is shown in both Cartesian and polar form. By resolving force into its *x* and *y* components, it is possible to find $\frac{d^2x}{dt^2} = -\frac{GM_s}{r^2}\cos\vartheta$ and $\frac{d^2y}{dt^2} = -\frac{GM_s}{r^2}\sin\vartheta$. By converting to polar coordinates, two new equations can be obtained: $x = r\cos\vartheta$ and $y = r\sin\vartheta$, and taking the second derivatives of these yields expressions for $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ in terms of *r* and ϑ:

$$\frac{d^2x}{dt^2} = -r\sin\vartheta\frac{d^2\vartheta}{dt^2} - r\cos\vartheta\left(\frac{d\vartheta}{dt}\right)^2 - 2\sin\vartheta\frac{dr}{dt}\frac{d\vartheta}{dt} + \cos\vartheta\frac{d^2r}{dt^2} \qquad \text{(Eq. 3)}$$

$$\frac{d^2y}{dt^2} = r\cos\vartheta\frac{d^2\vartheta}{dt^2} - r\sin\vartheta\left(\frac{d\vartheta}{dt}\right)^2 + 2\cos\vartheta\frac{dr}{dt}\frac{d\vartheta}{dt} + \sin\vartheta\frac{d^2r}{dt^2} \qquad \text{(Eq. 4)}$$

By manipulating and combining equations 3 and 4, two new simpler equations can be found:

$$-\frac{GM_s}{r^2} = -r\left(\frac{d\vartheta}{dt}\right)^2 + \frac{d^2r}{dt^2} \qquad \text{(Eq. 5)}$$

$$2\frac{dr}{dt}\frac{d\vartheta}{dt} = -r\frac{d^2\vartheta}{dt^2} \qquad \text{(Eq. 6)}$$

Replacing $\frac{d\vartheta}{dt}$ with a variable *p* in equation 6 allows $\frac{d\vartheta}{dt}$ to be written in terms of *r*.

$$2\frac{dr}{dt}p = -r\frac{dp}{dt}$$

When this is integrated, we obtain

$$\frac{d\vartheta}{dt} = \frac{h}{r^2}$$

Substituting this quantity into equation 5 results in the following equation:

$$-\frac{GM_s}{r^2} = \frac{-h^2}{r^3} + \frac{d^2r}{dt^2} \qquad \text{(Eq. 7)}$$

By setting a variable $u = \frac{1}{r}$, then

$$\frac{d^2r}{dt^2} = -GM_s u^2 + u^3h^2$$

Writing $\frac{dr}{dt}$ as follows makes it possible to find $\frac{dr}{dt}$ in terms of *h* and $\frac{du}{d\vartheta}$.

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\vartheta}\frac{d\vartheta}{dt}$$

$$\frac{dr}{dt} = -h\frac{du}{d\vartheta}$$

Differentiating the equation $\frac{dr}{dt} = -h\frac{du}{d\vartheta}$ with respect to ϑ and multiplying by $\frac{d\vartheta}{dt}$ shows that

$$\frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\vartheta^2}$$

When combined with equation 7 for $\frac{d^2r}{dt^2}$, it becomes clear that

$$\frac{d^2u}{d\vartheta^2} + u = \frac{GM_s}{h^2}$$

Solving this differential equation results in

$$u = A\cos\vartheta + B\sin\vartheta + \frac{GM_s}{h^2}$$

(A and B are simply constants in the above equation.) If we require that ϑ=0 when the Earth is closest to the sun, *r* will reach a minimum value and *u* will reach a maximum value when ϑ=0. Differentiating *u* shows that B=0; therefore, anything in the form $u = A\cos\vartheta + \frac{GM_s}{h^2}$ will satisfy the differential equation. The variable *u*, which is defined as the reciprocal of *r*, gives an expression for *r*:

$$r = \frac{\frac{h^2}{GM_s}}{A\frac{h^2}{GM_s}\cos\vartheta + 1}.$$

By letting the variable $e = \frac{Ah^2}{GM_s}$ (which we will find to be the eccentricity) the equation for *r* can be simplified to

$$r = \frac{\frac{e}{A}}{1 + e\cos\vartheta} \qquad \text{(Eq. 8)}$$

When ϑ is zero, the Earth will be closest to the sun, so *r* will reach a minimum. When ϑ is π, the Earth will be farthest from the sun, so *r* will reach a maximum. Therefore, $r_{\max} + r_{\min}$ will equal the length of the major axis, 2*a*.

$$r_{\min} = \frac{\frac{e}{A}}{1+e} \qquad r_{\max} = \frac{\frac{e}{A}}{1-e} \qquad r_{\max} + r_{\min} = 2a$$

When this equation is manipulated and A replaces its equivalent $\frac{eGM_s}{h^2}$, the final product is the equation $r = \frac{a(1-e^2)}{1+e\cos\vartheta}$, which, according to the work of the first group, is true of an ellipse with the sun at the focus. Therefore, the Earth moves in an elliptical path around the sun.

Kepler's Second Law

Kepler's Second Law states the relationship between the area swept out by the satellite and time.

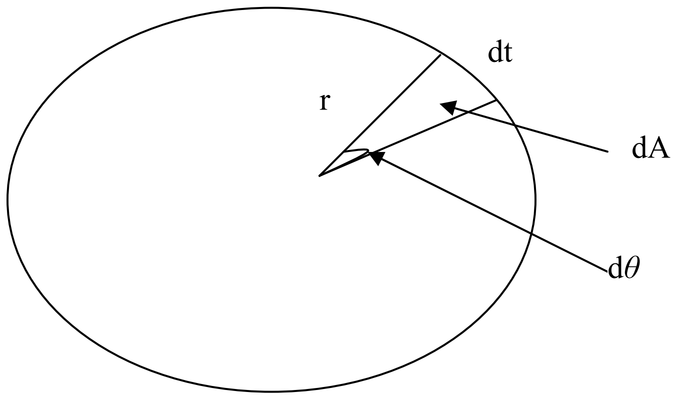

Figure 3 – Kepler's Second Law

The area of an ellipse can be found by integrating infinitesimally small sectors. These sectors can be approximated as sectors of a circle because when the area is infinitesimally small (dA) then the radius is constant. Using the formula $dA = \frac{r^2 d\vartheta}{2}$, the formula for the area of a sector of a circle, $\frac{dA}{dt}$ will eventually be shown to be equal to $\frac{h}{2}$.

Starting with chain rule, $\frac{dA}{dt} = \frac{dA}{d\vartheta}\frac{d\vartheta}{dt}$. Simple calculus shows that $\frac{dA}{d\vartheta} = \frac{r^2}{2}$. Remembering the value found for $\frac{d\vartheta}{dt}$ from Kepler's 1st law, $\frac{d\vartheta}{dt} = \frac{h}{r^2}$. Therefore, $\frac{dA}{dt} = \frac{r^2}{2}\frac{h}{r^2} = \frac{h}{2}$. Because h was defined in the previous equations as the constant of integration, $\frac{dA}{dt}$ is simply a constant. $\frac{dA}{dt}$ being a constant makes it very easy to find the area, which is simply $\frac{h}{2}$t.

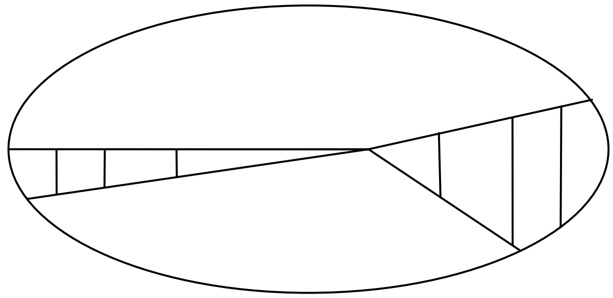

Figure 4 – Equal Areas in Equal Times

Because $\frac{dA}{dt}$ is a constant, the amount of area swept out by the Earth is always the same in a given amount of time. Thus, the Earth is moving more slowly when it is farther away from the sun, because equal areas must be swept out in equal time. So when the radius is longer, less distance must to be covered to sweep out the same area. This is illustrated in the figure above (Figure 4).

Kepler's Third Law

Having proven Kepler's Second Law, we can then prove his third, which states that the cube of the semi-major axis, *a*, of a planet's orbit is proportional to the square of the period of the orbit, T. Kepler's Second Law states that $\frac{dA}{dt} = \frac{h}{2}$, which leads to

$$A = \frac{ht}{2} \qquad \text{(Eq. 9)}$$

Since A is equal to the total area of the ellipse when time is equal to T (365.25 days for earth), we can use t=365.25 after substituting in a formula for the total area of the ellipse. The general formula for an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; therefore, $y = \frac{b}{a}\sqrt{a^2 - x^2}$. Integrating this gives the total area of the ellipse:

$$A = 2\frac{b}{a}\int_{-a}^{a}\sqrt{a^2 - x^2}\,dx = \pi ab$$

Using the area A, we find the period of the ellipse from Equation 9 to be $\pi ab = \frac{hT}{2}$, where T is the period of the orbit. Recall that $h = \sqrt{GM_S a(1-e^2)}$ and $e = \frac{c}{a}$. Making the necessary substitutions, we find that $T^2 = \frac{4\pi^2 a^2 b^2}{GM_S a(1-e^2)} = \frac{4\pi^2 a^3}{GM_S}$, which is Kepler's Third Law.

COMBINING KEPLER'S LAWS WITH ELLIPTICAL EQUATIONS TO PLOT THE PATH OF THE EARTH

The next step in the process is to combine the equations from the different groups to solve for the unknown quantities r and θ. We define a new variable M such that $M = E - e\sin E$, where e is the eccentricity of the Earth's orbit, and E is the angle between the x-axis and P′, as seen in Figure 1; previous equations are unsolvable without this substitution. By finding $\frac{dM}{dt}$, M can be found in terms of t, and therefore E can be found. After finding E, θ can be found, thereby allowing r to be found. Using these variables, we can plot the exact position of the Earth at any given time.

First, $\frac{dM}{dt}$ can be determined using the chain rule: $\frac{dM}{dt} = \frac{dM}{dE}\frac{dE}{d\theta}\frac{d\theta}{dt}$. Next, we can differentiate $M = E - e\sin E$ with respect to E:

$$\frac{dM}{dE} = 1 - \mathrm{ecos(E)} \qquad \text{(Eq. 10)}$$

Also, $\frac{dE}{d\theta}$ can be found by implicitly differentiating $\tan\frac{E}{2} = \sqrt{\frac{1-e}{e+1}}\tan\frac{\theta}{2}$.

$$\frac{dE}{d\theta} = \sqrt{\frac{1-e}{e+1}}\,\frac{\cos^2\frac{E}{2}}{\cos^2\frac{\theta}{2}} \qquad \text{(Eq. 11)}$$

Recalling that $\frac{d\vartheta}{dt} = \frac{h}{r^2}$ and substituting $\sqrt{GM_S a(1-e^2)}$ for h, it is seen that $\frac{d\vartheta}{dt} = \frac{\sqrt{GM_S a(1-e^2)}}{r^2}$. From the equation $a\cos E = r\cos\vartheta + ae$, r^2 can be substituted to show that

$$\frac{d\vartheta}{dt} = \frac{\sqrt{GM_S a(1-e^2)}}{a^2(\cos E - e)^2}\cos^2\vartheta \qquad \text{(Eq. 12)}$$

By multiplying equations 10, 11, and 12, $\frac{dM}{dt}$ can be found:

$$\frac{dM}{dt} = \frac{\sqrt{GM_S}}{a^{\frac{3}{2}}}$$

Kepler's Third Law of planetary motion states that

$$T^2 = \frac{4\pi^2 a^3}{GM_S}$$

This allows $\frac{dM}{dt}=\frac{\sqrt{GM_S}}{a^{\frac{3}{2}}}$ to be written as $\frac{dM}{dt}=\frac{2\pi}{T}$. Integrating this expression with respect to *t*, and allowing the period to be 365.25 days, $M=\frac{2\pi}{365.25}t$, where *t* is the time in days since perihelion (January 4, 18 hours GMT for the year 2004) (Earth's Seasons, n. pag.). Replacing M with its equivalent allows *E* to be determined for any time *t*:

$$M=E-e\sin E=\frac{2\pi}{365.25}t$$

It is impossible to solve the equation *M*=*E*-esin*E* for *E*; therefore, a series of approximations must be used. The eccentricity of the Earth's orbit, *e*, is very small (about 0.0167), and when multiplied by sin*E*, which is always less than or equal to one, *e*sin*E* will be even smaller (Williams, n. pag.). Thus, *E* can be initially approximated as *M*. Therefore, the first approximation of *E* is $E_1=M$.

Now, by substituting E_1 into the original equation, *M*=*E*-*e*sin*E*, the second approximation can be found.

$$E_2=M+e\sin E_1$$

By repeating this step, the third approximation can be found, and the fourth, and so on.

$$E_n=M+e\sin E_{n-1}$$

By using the equation for *M*, $M=\frac{2\pi}{365.25}t$, and the above approximations for *E*, *E* can be found for any time. For example, when it is August 1st at 12:00 midnight, the time in days is 208.75 since perihelion. At this point in time, *E* is about 3.584 radians.

By using equation 2

$$\tan\frac{E}{2}=\sqrt{\frac{1-e}{1+e}}\tan\frac{\vartheta}{2}$$

ϑ is found to be about -2.706 radians, or its positive equivalent, 3.577 radians. This same calculation can easily be performed for any date and time. To translate ϑ and *r* into observable quantities for viewers on Earth it is necessary to examine the celestial sphere.

SPHERICAL TRIGONOMETRY

Derivation of Law of Cosines for Spherical Trigonometry:

When dealing with celestial mechanics it is necessary to work with spheres. Looking into the sky is comparable to looking at the interior of a giant sphere; each object appears to be the same distance away. When determining the timing of events, regular trigonometry does not suffice. In order to make any predictions, the proper equations must be derived. To equate distances between objects and the time objects pass into sight, we can use spherical triangles.

A spherical triangle is defined by its arcs upon the surface of the sphere. Like a planar triangle it has three sides joined together at vertices. The angles are measured as the tangential rays to the surface arcs that stem from the vertex. The arcs are all parts of great circles, which are circles upon the surface of the sphere whose centers are also the center of the sphere. Unlike planar triangles, spherical ones do not have interior angle sums of 180°. With all of this information taken into account, we can find an equivalent law of cosines for the spherical triangle.

To that end, it is necessary to construct a geometric model to aid in finding the law of cosines. This model consists of a sphere sitting on a plane. It is used to relate the laws of planar trigonometry to spherical trigonometry. Three points are created on the sphere—the three points of a spherical triangle. Of the three points, one is positioned so that it touches the plane. Then, from the center of the sphere, two rays are drawn through the two selected points on the sphere, eventually intersecting two more points on the plane. These two new points can be connected with the point where the sphere touches the plane, and with each other. This creates the planar triangle shown in Figure 5 below.

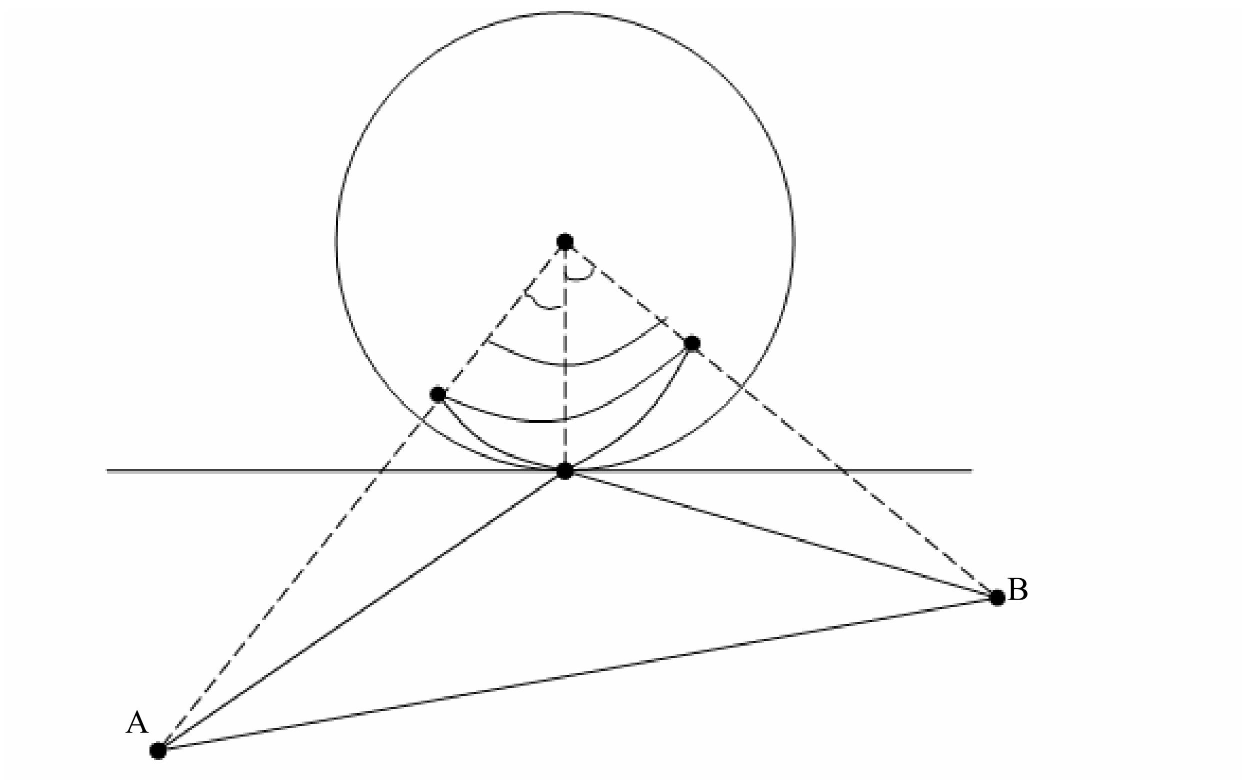

Figure 5-Spherical Triangle on a Plane

Given the radius of the sphere, the lengths of three arcs of the spherical triangle and its angles, we can create a spherical law of cosines similar in format to the planar law of cosines. To facilitate this process we use the planar triangle A'B'C and then transform the information back to fit the spherical triangle ABC.

With this information we derive the following:

i) $b' = r\tan\frac{b}{r}$ $\quad a' = r\tan\frac{a}{r}$ $\qquad c'^2 = a'^2 + b'^2 - 2a'b'\cos C$ (law of cosines)

Law of cosines with substitution: $c' = \sqrt{r^2\tan^2\frac{a}{r} + r^2\tan\frac{b}{r} - 2r^2\tan\frac{a}{r}\tan\frac{b}{r}\cos C}$

The cosine law is then derived in a different way through trigonometry:

ii) OB'= $\dfrac{r}{\cos\frac{a}{r}}$ OA'= $\dfrac{r}{\cos\frac{b}{r}}$ $\qquad c' = \sqrt{\dfrac{r^2}{\cos^2\frac{a}{r}} + \dfrac{r^2}{\cos^2\frac{b}{r}} - 2\dfrac{r}{\cos\frac{a}{r}}\dfrac{r}{\cos\frac{b}{r}}\cos\frac{c}{r}}$

After equating the 2 expressions for c', the law of cosines is found through reduction and equation manipulation.

iii) $c'^2 = c'^2$

$$r^2\tan^2\frac{a}{r} + r^2\tan\frac{b}{r} - 2r^2\tan\frac{a}{r}\tan\frac{b}{r}\cos C = r^2\sec^2\frac{a}{r} + r^2\sec^2\frac{b}{r} - 2r^2\sec\frac{a}{r}\sec\frac{b}{r}\cos\frac{c}{r}$$

$$\therefore \cos\frac{c}{r} = \cos\frac{a}{r}\cos\frac{b}{r} + \sin\frac{a}{r}\sin\frac{b}{r}\cos C \qquad \text{(Eq. 13)}$$

Derivation of Law of Sines for Spherical Trigonometry

Once the law of cosines for spherical triangles is derived, a law of sines for spherical triangles is needed to complete our work on calculating the components of a spherical triangle. Like the derivation of the law of cosines for spherical triangles, in which planar trigonometry is used, the derivation of the law of sines for spherical triangles also requires information from planar trigonometry. Noticing that the law of sines for planar triangles can be derived from the law of cosines for planar triangles, we are able to use the same techniques applied to the spherical law of cosines to derive the spherical law of sines. Recall from Equation 13 that, when written in terms of $\frac{c}{r}$, the law of cosines for spherical trigonometry states:

$$\cos\frac{c}{r} = \cos\frac{a}{r}\cos\frac{b}{r} + \sin\frac{a}{r}\sin\frac{b}{r}\cos C$$

The law of cosines is then solved in terms of cos$C$. This new form of the spherical law of cosines is then squared and written in terms of $\cos^2 C$. Then, using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, the equation is solved for $\sin^2 C$, as follows:

$$\sin^2 C = \frac{\sin^2\frac{a}{r}\sin^2\frac{b}{r} - \cos^2\frac{c}{r} + 2\cos\frac{a}{r}\cos\frac{b}{r}\cos\frac{c}{r} - \cos^2\frac{a}{r}\cos^2\frac{b}{r}}{\sin^2\frac{a}{r}\sin^2\frac{b}{r}}$$

This equation as solved for $2\cos\frac{a}{r}\cos\frac{b}{r}\cos\frac{c}{r}$ is:

$$2\cos\frac{a}{r}\cos\frac{b}{r}\cos\frac{c}{r} = \sin^2 C\sin^2\frac{a}{r}\sin^2\frac{b}{r} + \cos^2\frac{a}{r}\cos^2\frac{b}{r} + \cos^2\frac{c}{r} - \sin^2\frac{a}{r}\sin^2\frac{b}{r}$$

This derivation is repeated with the law of cosines in terms of $\frac{b}{r}$, instead of $\frac{c}{r}$. Thus, the starting equation of the second derivation is:

$$\cos\frac{b}{r} = \cos\frac{a}{r}\cos\frac{c}{r} + \sin\frac{a}{r}\sin\frac{c}{r}\cos B$$

When the derivation is repeated it leads to:

$$2\cos\frac{a}{r}\cos\frac{b}{r}\cos\frac{c}{r} = \sin^2 B\sin^2\frac{a}{r}\sin^2\frac{c}{r} + \cos^2\frac{a}{r}\cos^2\frac{c}{r} + \cos^2\frac{b}{r} - \sin^2\frac{a}{r}\sin^2\frac{c}{r}$$

Because both equations equal $2\cos\frac{a}{r}\cos\frac{b}{r}\cos\frac{c}{r}$, the right sides of the two equations are equal to each other. Therefore:

$$\sin^2 B\sin^2\frac{a}{r}\sin^2\frac{c}{r} + \cos^2\frac{a}{r}\cos^2\frac{c}{r} + \cos^2\frac{b}{r} - \sin^2\frac{a}{r}\sin^2\frac{c}{r} =$$
$$\sin^2 C\sin^2\frac{a}{r}\sin^2\frac{b}{r} + \cos^2\frac{a}{r}\cos^2\frac{b}{r} + \cos^2\frac{c}{r} - \sin^2\frac{a}{r}\sin^2\frac{b}{r}$$

After simplifying this equation through factoring and the trigonometric identity, $\cos^2\theta + \sin^2\theta = 1$, it becomes:

$$\sin^2\frac{c}{r}\left(\sin^2 B - 1\right) - \sin^2\frac{b}{r}\left(\sin^2 C - 1\right) - \left(\cos^2\frac{c}{r} - \cos^2\frac{b}{r}\right) = 0$$

Further simplification of this equation through distribution leads to:

$$\sin^2\frac{c}{r}\sin^2 B - \sin^2\frac{b}{r}\sin^2 C = 0$$

After moving $\sin^2\frac{b}{r}\sin^2 C$ to the other side of the equation and simplifying, the result is:

$$\frac{\sin B}{\sin\frac{b}{r}} = \frac{\sin C}{\sin\frac{c}{r}}$$

We can then repeat this process with the law of cosines in terms of $\frac{a}{r}$ and come to the same conclusion. This leads to the law of sines for spherical triangles, which is:

$$\frac{\sin A}{\sin\frac{a}{r}} = \frac{\sin B}{\sin\frac{b}{r}} = \frac{\sin C}{\sin\frac{c}{r}} \qquad \text{(Eq. 14)}$$

Application to the Celestial Sphere

After deriving the law of sines and law of cosines for a spherical triangle, we can apply these concepts to the celestial sphere. The celestial sphere is an imaginary sphere of infinite radius with the Earth at its center, the view of space as seen from Earth. All objects in the sky are assumed to lie on the surface of the celestial sphere. In order to determine the location of objects on the celestial sphere, astronomers use a coordinate system similar to latitude and longitude: declination (δ) and right ascension (α). We must therefore derive equations that can be used to calculate the declination and right ascension of any object ($*$) located on the celestial sphere given the value of λ (the distance in degrees from the vernal equinox, the location of the sun on the celestial sphere on the first day of spring, to the object) and the value of ε (the degree tilt of the Earth on its axis).

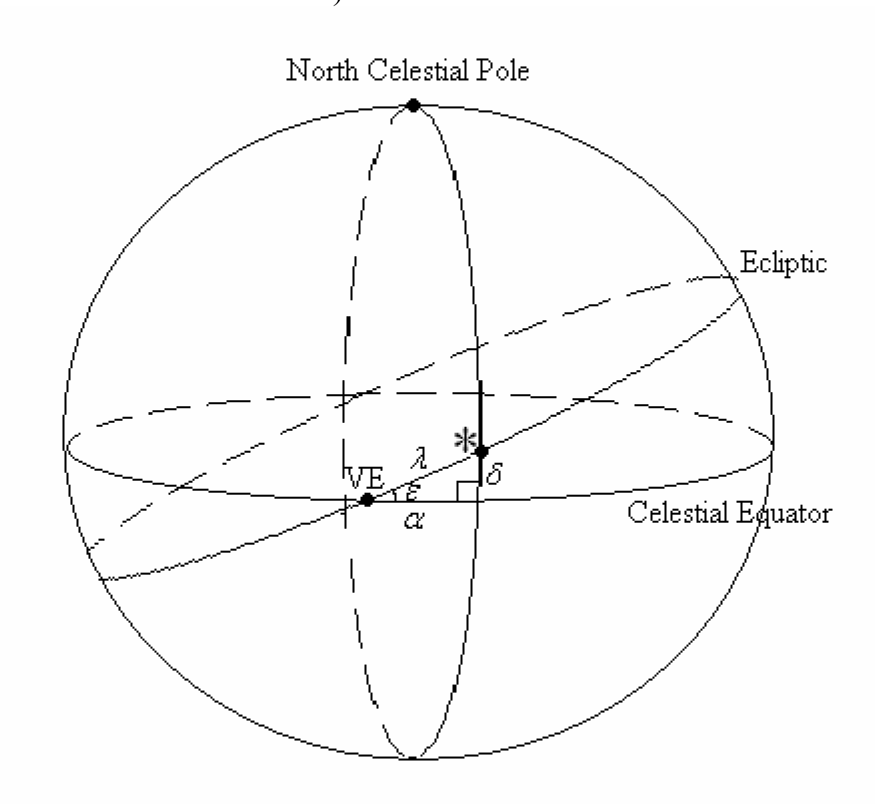

Figure 6 – Celestial Sky

Using the law of cosines we previously derived, we can create an equation for α (right ascension) using the triangle formed by the three values $\lambda, \delta, \text{and}\, \alpha$:

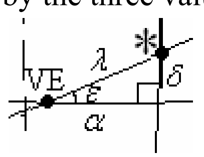

Figure 7 – Closeup of above triangle

$$\cos\lambda = \cos\alpha\cos\delta + \sin\alpha\sin\delta\cos 90^{\circ}$$
$$\cos\alpha = \frac{\cos\lambda}{\cos\delta}$$

Our next task is to use the same two given values (λ and ε) to find an equation for δ (declination). We begin by applying the law of sines for spherical triangles to the triangle in Figure 8 below.

$$\sin\lambda = \frac{\sin\delta}{\sin\varepsilon}$$
$$\sin\delta = \sin\lambda\sin\varepsilon$$

Application to Elliptical Motion

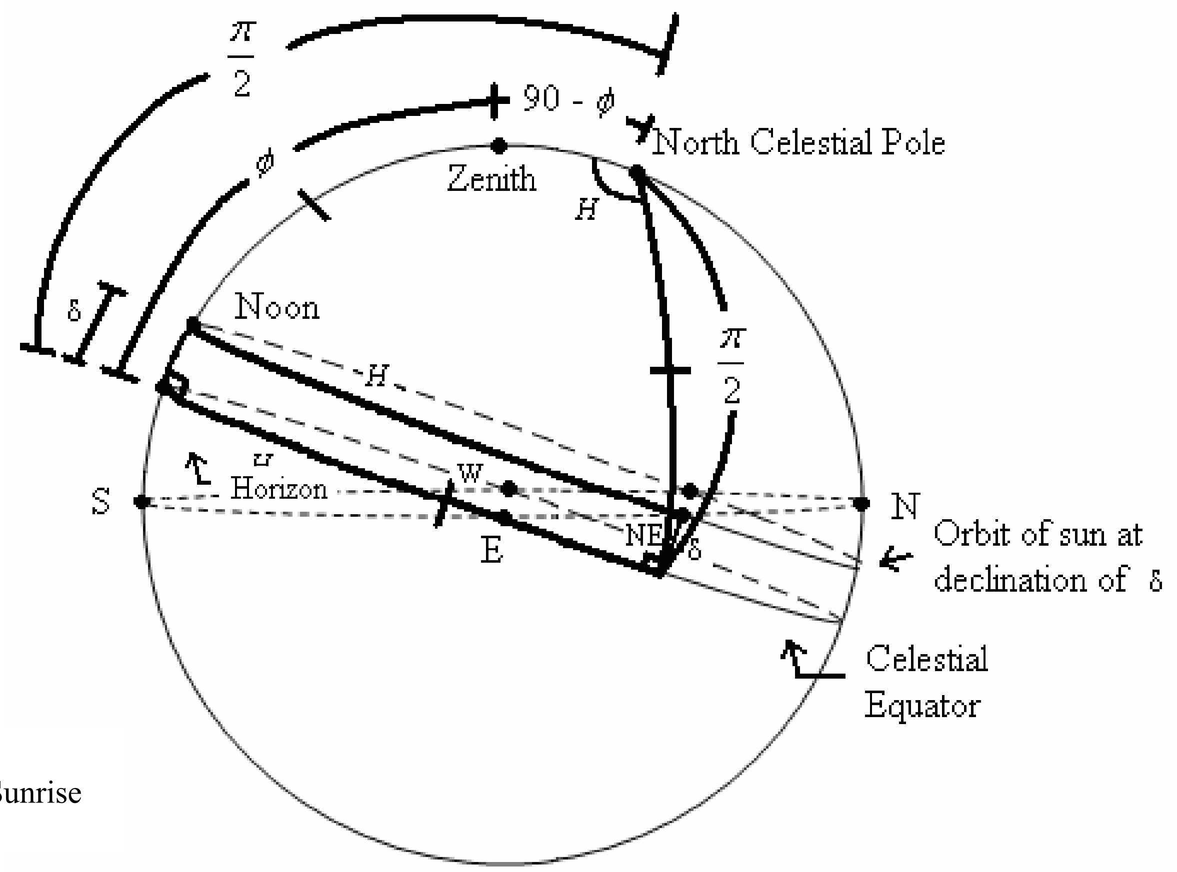

Figure 8 – Calculating Sunrise and Sunset

The derived equations and formulas only apply to spherical triangles that have sides made of arcs of great circles. Therefore, when trying to determine the length of *H* in Figure 8, the number of hours from sunrise to noon (the highest point of the sun) when the sun's orbit has a declination of δ, we must first find the length of *H* on a great circle. To accomplish this we bring down an arc perpendicular to both circles and the length of the resulting segment is δ. Then, with an arc length of *H* translated onto a great circle, we can form a spherical triangle by making the other two arcs connect to the North Celestial Pole (this spherical triangle is signified by the three arcs with tick-marks). It can now be determined that the two bottom angles are right angles and that the top angle is *H,* because it intercepts an arc length of *H*. Thus, we have enough information to substitute our findings into the previously derived law of cosines for spherical triangles ($\cos\frac{c}{r} = \cos\frac{a}{r}\cos\frac{b}{r} + \sin\frac{a}{r}\sin\frac{b}{r}\cos C$) to find *H*:

$$\cos\frac{\pi}{2} = \cos\left(\frac{\pi}{2}-\phi\right)\cos\left(\frac{\pi}{2}-\delta\right) + \sin\left(\frac{\pi}{2}-\phi\right)\sin\left(\frac{\pi}{2}-\delta\right)\cos H$$

$$\cos H = -\tan\phi\tan\delta$$

Next, in order to account for the Earth's orbit around the sun in addition to its rotation about its own axis, we can use the following diagram and discern an equation to find λ:

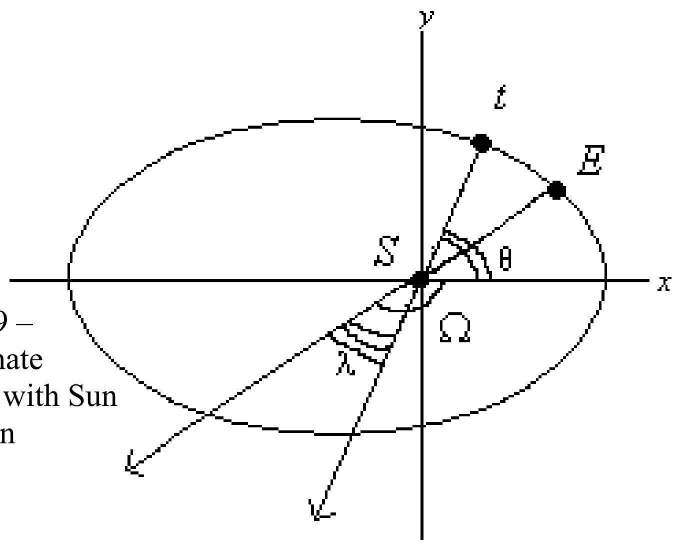

Figure 9 – Coordinate System with Sun at Origin

$$\vartheta + \Omega - \lambda = \pi$$
$$\lambda = \vartheta + \Omega - \pi$$

In Figure 9, ϑ is the angle formed by the Earth, sun, and a point on the x-axis known as the perihelion (Earth's closest approach to the sun). Thus, after deriving ϑ, we can pinpoint Earth's exact location on its orbit around the sun. The point designated by *t* is the Earth's point in its orbit on the first day of spring, the vernal equinox. If we construct a line that passes through this point and the sun, we obtain a universal reference line where λ = 0. The angle of perihelion from the universal reference line to the vernal equinox is a constant, Ω, known as the argument of the perihelion, which is approximately 1.59577 radians in measure. We can find λ with the above equation and substitute it into the previously derived equations to find α and δ, the right ascension and declination, respectively.

After obtaining a value of ϑ for August 1, 2004, we are able to solve for λ, α, and δ. Then, given ϕ (latitude of the observer, which is the distance from the observer to the zenith), we obtain *H*, half the number of hours of daylight. We subtract *H* from and add *H* to 1:00 PM (noon during daylight savings time) to find the exact times of sunrise and sunset on August 1, which are 5:54 AM and 8:06 PM in Madison, New Jersey. With the derived equations, we are able to find the exact times of sunrise and sunset on any given day at any given location on Earth.

MAKING A SUNDIAL

A sundial can be created using much of the same the type of mathematics as was used to model the paths of planets. During the course of the day, the sun follows a well-defined circular path across the sky that is a distance *r* from the Earth. To make a sundial, a “stick” pointing due north (in the northern hemisphere) has to be inserted into the ground at the same angle as its location’s latitude (in this case, 40° 42’). Because we are in daylight savings time, the sun is at its highest point at 1 PM and at this time the shadow of the stick is directly under the stick. However, at 12 pm, the sun has progressed 15 fewer degrees in its path. Drawing a three-dimensional coordinate system helps to illustrate this scenario (Figure 10).

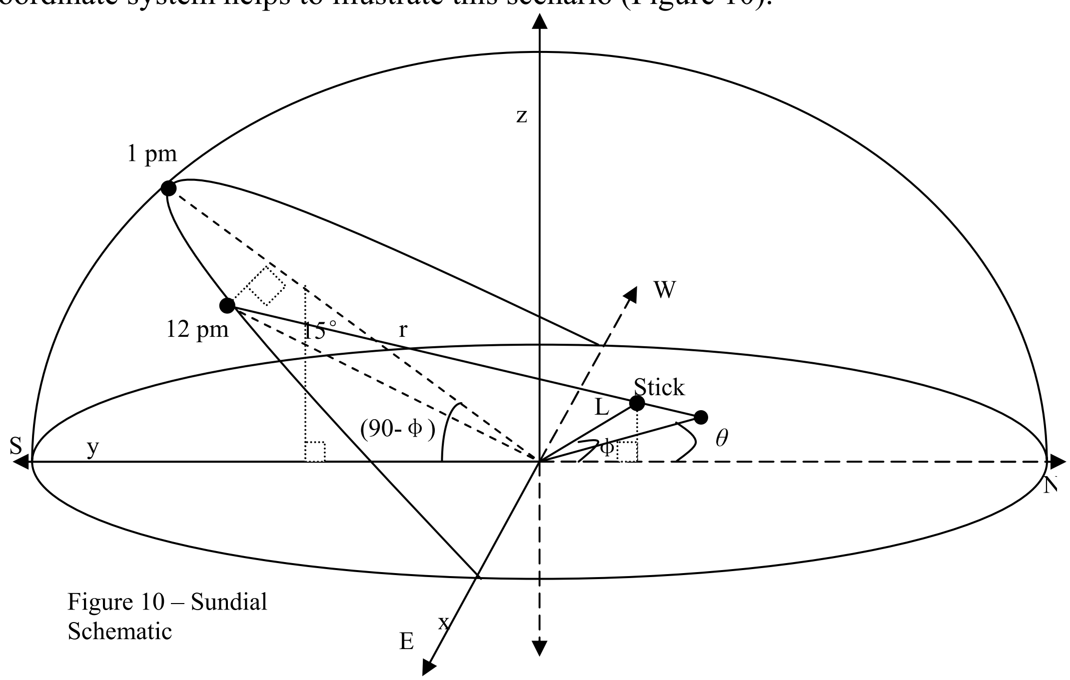

Figure 10 – Sundial Schematic

The objective for this part of the project is to determine the coordinates of the shadow of the point of the stick at 12 PM. To do this, we begin by finding the coordinates of both the sun and the stick. The coordinates at the tip of the stick are $(0, L\sin\phi, -L\cos\phi)$.

The sun’s coordinates are more complex. Using the dotted lines that form right triangles in Figure 10, we observe the sun’s coordinates to be $(r\sin 15, r\cos 15\sin\phi, r\cos 15\cos\phi)$.

Given two coordinates in three-dimensional space, it is possible to create a line that runs through the two points by using vector notation. If (x_1,y_1,z_1) and (x_2,y_2,z_2) are the two points, then the line connecting them is of the form

$$((x_2 - x_1)t + x_1)\mathbf{i} + ((y_2 - y_1)t + y_1)\mathbf{j} + ((z_2 - z_1)t + z_1)\mathbf{k}$$

where **i**, **j**, and **k** are the unit vectors in the x, y, and z directions, respectively. Substituting the given coordinates for the sun and the end of the stick gives us the line *l* through the two points: $((r\sin 15°)t + 0)$**i** + $[(r\cos 15° \sin\phi - L\cos\phi)t + L\cos\phi]$**j** + $[(r\cos 15° \cos\phi - L\sin\phi)t + L\sin\phi]$**k**.

Because we want the coordinates where the shadow hits the ground, we want *t* such that z=0. Using the **k**-vector, we find that this occurs when $t = \dfrac{-L\sin\phi}{r\cos 15° \cos\phi - L\sin\phi}$.

Plugging this value of *t* into the equation of the line gives

$$x = \frac{-r\sin 15° L\sin\phi}{r\cos 15° \cos\phi - L\sin\phi} \text{ and } y = ((r\cos 15° - L\cos\phi)\frac{-L\sin\phi}{r\cos 15° \cos\phi - L\sin\phi} + L\cos\phi).$$

Let θ be the angle between the shadow of the stick at 1 PM, and the shadow at noon. The tangent of θ is equal to $\frac{x}{y}$ and, after plugging in the given values of x and y, it is found that $\tan\theta = \sin\phi \tan 15°$. Using this information, it is possible to create a sundial, replacing 15° with 30° or 45° to obtain the angles for different times of the day.

CONCLUSION

In this team project, we were given the task of mathematically determining the equations that govern the motion of satellites, focusing on the Earth revolving around the sun. In order to explore the different aspects of this complex motion effectively, it was necessary to divide the work into sections with complementary goals. One group derived cosine and sine laws for spherical triangles, another group proved Kepler's three laws of motion by using calculus, and the last group explored the properties of an ellipse, the type of path the Earth takes around the sun. One final product of our work was the derivation of equations that were used to calculate the time of sunrise and sunset on any given day in the year; we also figured out how to construct a sundial.

In the spherical trigonometry group, the rules of planar trigonometry were used to find equivalent sine and cosine laws for spherical triangles. Writing the regular law of cosines in terms of given values led to two equations that we used to derive the spherical cosine law. We worked from the spherical cosine law to find the sine law for triangles drawn on spheres; this was then applied to the celestial sphere with right ascension and declination. Then, with information from the other two groups, it was possible to determine the time of day break on the morning of the 1st of August.

While one of our groups worked with spherical trigonometry, another had the task of proving Kepler's laws of planetary motion using calculus—an important step in understanding the math behind the motion of satellites. We started with Newton's law of gravity in polar form, and after much algebraic manipulation, were able to prove the three laws of Kepler.

Our last team worked on the math behind ellipses; the first step this group took was to convert the Cartesian equation for an ellipse into polar form. We next found the relation between a circle and an inscribed ellipse, which was essential to the understanding of elliptical motion in terms of equations.

Our Kepler's law group and elliptical geometry group arrived at the same mathematical equation for the Earth's orbit. We then used our combined equations to find the location of Earth in its orbit on August 1, 2004. This information, in turn, allowed the spherical trigonometry group to calculate the sunrise time for August 1, 2004.

The final task was to apply mathematical and physical principles to construct a sundial. The angles for the sundial were calculated through right triangle trigonometric relationships. The direction of the shadow of a stick in the ground is directly north at one o'clock in the afternoon due to daylight savings time. This shadow created the reference line. We then used meter sticks, T-squares, and simple trigonometry to plot the angles for the different hours of the day. Finally, we erected a pole at a 41º angle, since our region is at approximately 41º latitude. This situation called for a little more trigonometry. We supported the pole with a brace, and the shadow from the pole indicated the time.

In this project we gained an understanding of the principles of celestial mechanics. Because the Earth rotates, the Earth can be depicted as the center of a celestial sphere with the rest of the universe on the sphere's surface. Also, because the Earth revolves about the sun, the Earth can be described as a satellite traveling in an elliptical path. The construction of a sundial and the derivation of Kepler's laws verify this model of the Earth's motion. Using our knowledge of mathematical physics, not only can we find the sunrise and sunset on Earth and the Earth's orbital motion, but the same model can be applied to any satellite revolving about the sun.

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