# CELESTIAL MECHANICS

Andrew Davis, Mary Germino, Ethan Groveman, Semmie Kim,
Daniel Morgan, Kate Safin, Jake Snell, Alexander Stephan,
Denys Voytenko, Rosalie Yan, Daniel Yoo, Eileen Zhuang

Advisor: Steve Surace
Assistant: Margaret Senese

## ABSTRACT

The purpose of this team project was to investigate the motion of celestial bodies. First, we derived Kepler's Laws. We were then able to calculate a planet's angle in orbit in relation to its perihelion by translating an ellipse, expressed in polar coordinates, and constructing an auxiliary circle. By introducing spherical trigonometry, we were able to track a body's movement on the celestial sphere. In addition, we used trigonometry to calculate the angles of a rotation of a shadow at different times of day, and constructed a sundial. Finally, we successfully unified our studies of Kepler's Laws, the geometry of ellipses, and spherical trigonometry to pinpoint the position of the Sun and planets on the celestial sphere at any given time. Our findings could be further applied to the movement of other celestial bodies.

## INTRODUCTION

In 1618, after years of observation and careful formulation, the crowning achievement of Johannes Kepler was complete. The three laws of planetary motion, revolutionary in their new elliptical view of the solar system, suggested a radically new yet unproven view of planetary motion. The appearance of Newton's gravity laws and system of calculus marked down in theory what was once postulated years before. The cosmos, held together by the universal Newtonian gravity laws, was conquered by a new physics.

The aim of this team project was to follow and reassert what was and is an incredibly important conjunction of observation and theory. First, it was necessary to derive Kepler's Laws. The study of elliptical geometry then allowed us to make practical use of the equations we had derived. Through the determination of a law of cosines and a law of sines for spherical triangles, we were able to model a planet's motion on the celestial sphere. We also made a sundial.

## PART I: KEPLER'S LAWS

### Kepler's First Law

Kepler's First Law allows one to calculate the distance $r$ between a planet and the star it orbits given $\theta$, the angle between the major axis of the planet's orbit and the line connecting the planet and the star. We begin with a diagram of the planet and a star (Fig. 1).

A planet of mass $m$ experiences a gravitational force $F$ towards a star of mass $M$. The x-axis is defined to be the major axis of the planet's orbit. The center of the star is located at the origin. Let the x-component of $F$ be $F_x$ and the y-component of $F$ be $F_y$.

Using Newton's Law of Universal Gravitation:
$F = -\frac{GMm}{r^2}$ (where $G$ is the gravitational constant), and Newton's Second Law of Motion, $F = ma$, we can express $F_x$ and $F_y$ in two different ways:

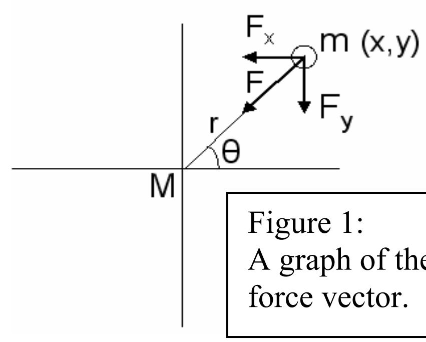



Figure 1:
A graph of the force vector.

$$F_x = \frac{-GMm}{r^2}\cos\theta = m\frac{d^2x}{dt^2},$$
$$F_y = \frac{-GMm}{r^2}\sin\theta = m\frac{d^2y}{dt^2}.$$

(Note that $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ are the accelerations of the planet in the x and y directions.)

From the diagram, we see that $x = r\cos\theta$ and $y = r\sin\theta$.
Now we substitute these values in to get

$$F_x = \frac{-GMm}{r^2}\cos\theta = m\frac{d^2}{dt^2}r\cos\theta, \tag{1}$$
$$F_y = \frac{-GMm}{r^2}\sin\theta = m\frac{d^2}{dt^2}r\sin\theta. \tag{2}$$

Finding the second derivative of $r\cos\theta$, you obtain

$$\frac{d^2}{dt^2}r\cos\theta = -\left(\frac{d^2\theta}{dt^2}\right)r\sin\theta + \left(\frac{d^2r}{dt^2}\right)\cos\theta - \left(\frac{d\theta}{dt}\right)^2 r\cos\theta - 2\frac{dr}{dt}\frac{d\theta}{dt}\sin\theta. \tag{3}$$

Likewise, the second derivative of $r\sin\theta$ is

$$\frac{d^2}{dt^2}r\sin\theta = \left(\frac{d^2\theta}{dt^2}\right)r\cos\theta + \left(\frac{d^2r}{dt^2}\right)\sin\theta + 2\frac{dr}{dt}\frac{d\theta}{dt}\cos\theta - \left(\frac{d\theta}{dt}\right)^2 r\sin\theta. \tag{4}$$

Insert the second derivatives (3, 4) into equations (1) and (2) to obtain

$$\frac{-GM}{r^2}\cos\theta = -\left(\frac{d^2\theta}{dt^2}\right)r\sin\theta + \left(\frac{d^2r}{dt^2}\right)\cos\theta - \left(\frac{d\theta}{dt}\right)^2 r\cos\theta - 2\frac{dr}{dt}\frac{d\theta}{dt}\sin\theta, \tag{5}$$
$$\frac{-GM}{r^2}\sin\theta = \left(\frac{d^2\theta}{dt^2}\right)r\cos\theta + \frac{d^2r}{dt^2}\sin\theta + 2\frac{dr}{dt}\frac{d\theta}{dt}\cos\theta - \left(\frac{d\theta}{dt}\right)^2 r\sin\theta. \tag{6}$$

Multiply (5) by $\sin\theta$, multiply (6) by $-\cos\theta$ and add the resulting equations to obtain:

$$0 = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}. \tag{7}$$

Multiply (5) by $\cos\theta$, multiply (6) by $\sin\theta$ and add the resulting equations to obtain

$$\frac{-GM}{r^2} = -r\left(\frac{d\theta}{dt}\right)^2 + \frac{d^2r}{dt^2}. \tag{8}$$

To make calculations easier, we let $p = \frac{d\theta}{dt}$ and substitute this into (7). After algebraic manipulation, we get

$$-r\frac{dp}{dt} = 2p\frac{dr}{dt} \Rightarrow -\frac{dp}{p} = 2\frac{dr}{r}.$$

We integrate each side where the constant of integration is $h$ to get

$$p = \frac{d\theta}{dt} = \frac{h}{r^2}. \tag{9}$$

Substituting this back into (8) and simplifying, we obtain

$$r = \frac{\frac{h^2}{GM}}{\frac{Ah^2}{GM}\sin\theta + \frac{Bh^2}{GM}\cos\theta + 1}. \tag{10}$$

Therefore, to obtain the smallest value of $r$, which we have defined to occur when $\theta = 0$, we maximize $A\sin\theta + B\cos\theta + \frac{GM}{h^2}$.

To maximize $A\sin\theta + B\cos\theta + \frac{GM}{h^2}$, we take the derivative with respect to θ to obtain any critical points and get

$$A\cos\theta - B\sin\theta.$$

Now we equate this to 0 and obtain the equation

$$A\cos\theta = B\sin\theta.$$

Substituting $\theta = 0$, we find that A = 0. Now, our equation (10) becomes

$$r = \frac{\frac{h^2}{GM}}{\frac{Bh^2}{GM}\cos\theta + 1},$$

the equation of an ellipse in polar form.

Kepler's Second Law

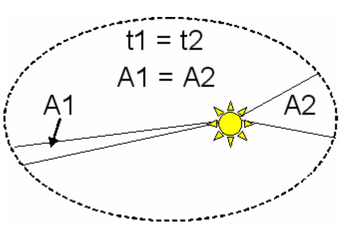



Fig. 2 : Kepler's Second Law. A1=A2

A line joining a planet and its star sweeps out equal areas during equal intervals of time. If a planet is closer to the sun, it will orbit faster, and if it is further from the sun, it will orbit slower. As you can see in Fig. 2, Area 1 and Area 2 are equal in size.

In order to derive this law, we must show that the change in area for some change in time is not dependent on any other variables. We start with the circular equation for area, approximating the ellipse as a circle.

After manipulation, it becomes

$$dA = \frac{d\theta}{2\pi}\pi r^2 ,$$

$$\frac{d\theta}{dt}\frac{r^2}{2} = \frac{dA}{dt} .$$

From our earlier calculations, we can recall the following substitution (9) that can be used to further simplify the equation

$$\frac{d\theta}{dt} = \frac{h}{r^2}$$

to

$$\frac{h}{2} = \frac{dA}{dt} . \qquad (11)$$

Since h is a constant, $\frac{h}{2}$ is a constant, and $\frac{dA}{dt}$ is a constant, the area swept out by the earth in its orbit around the sun is equal during equal intervals of time.

Kepler's Third Law

Kepler's Third Law describes the relationship between the period (T) of a planet and the semimajor axis of its orbit (a). Deriving it produces an equation relating $T^2$ and $a^3$.

We start with $\frac{dA}{dt}$, which is a constant according to Kepler's Second Law (11):

$$\frac{dA}{dt} = \frac{h}{2} .$$

We integrate $\frac{dA}{dt}$ from 0 to T to represent the area of the elliptical orbit of a planet:

$$\int_0^A \frac{dA}{dt} = \int_0^T \frac{h}{2} ,$$

$$A = \frac{h}{2}T \cdot$$

We also know that the area of an ellipse can be expressed as

$$A = \pi ab ,$$

$$\pi ab = \frac{h}{2}T \cdot$$

We calculate an equivalent expression for b:

$$a^2 + b^2 = c^2$$

$$b^2 = c^2 - a^2$$

$$= a^2(1 - (c/a)^2)$$

Because $e = \frac{c}{a}$,

$$b^2 = a^2(1 - e^2) ,$$

$$b = a \cdot \sqrt{1-e^2}\ .$$

Then, because
$$h = \sqrt{GM \cdot a \cdot (1-e^2)}\ , \tag{12}$$

$$b = \frac{h\sqrt{a}}{\sqrt{GM}}\ .$$

Substituting in for h, we get

$$b = \frac{\sqrt{GM \cdot a \cdot (1-e^2)} \cdot \sqrt{a}}{\sqrt{GM}}\ .$$

We then substitute this in for b and get

$$\pi a \cdot a \cdot \sqrt{1-e^2} = \frac{h}{2} T\ .$$

Substituting for h, we get

$$\pi a^2 \sqrt{1-e^2} = \frac{\sqrt{GM \cdot a \cdot (1-e^2)}}{2} T\ ,$$

$$2\pi a^2 = \sqrt{GM \cdot a} \cdot T$$

$$4\pi^2 a^4 = GM \cdot a \cdot T^2$$

$$\frac{4\pi^2}{GM} \cdot a^3 = T^2\ . \tag{13}$$

Therefore, $T^2$ and $a^3$ are related by a factor of $4\pi^2$ / GM.

**PART II: STUDYING ELLIPTICAL GEOMETRY**

For this part of the team project, we needed to study the patterns and equations associated with an ellipse, to better understand the processes of planetary motion. This knowledge eventually helps to unlock the practical applications of Kepler's Laws for the greater goal of trying to pinpoint the exact locations of the planets at any time.

Ellipses as Functions of Polar Coordinates

Before we begin to uncover any patterns involved with the motion of the planets, it should first be mentioned that for all of the planets' orbits, the sun lies on one focus of their elliptical orbits. It is easiest to continue discussing planetary motion by placing the sun at the origin on a standard xy-coordinate plane. In order to do so, we need to shift the ellipse (Fig. 3) to the left by c, the length of one focus from the origin:

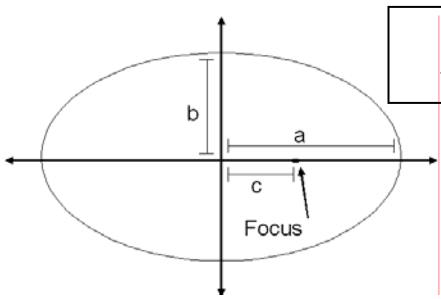



Fig. 3: An Ellipse

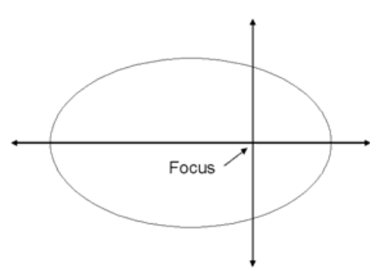



Fig. 4: An Ellipse centered at a focus

$$\frac{\left(x+\sqrt{a^2-b^2}\right)^2}{a^2}-\frac{y^2}{b^2}=1 \tag{14}$$

Now, the focus at the origin (Fig. 4) represents the position of the sun, with the path of a planet's orbit around it. This shift allows us to simplify the equation for the ellipse by converting it into polar-coordinates (Fig. 5). *r* will represent the distance from the sun to the planet and θ will represent the angle that the planet makes with the sun and the x-axis, respectively. (14) becomes

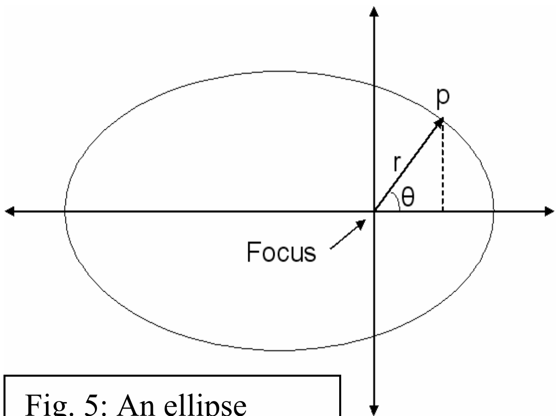



Fig. 5: An ellipse defined in polar form

$$\frac{\left(x+\sqrt{a^2-b^2}\right)^2}{a^2}+\frac{y^2}{b^2}=1,$$

$$b^2x^2+a^2y^2=b^4-2b^2x\sqrt{a^2-b^2},$$

$$b^2x^2+a^2y^2=b^4-2b^2cx$$

Now we refer to the polar-coordinate definitions ($x=r\cos\theta; y=r\sin\theta$) and substitute for x and y:

$$b^2r^2\cos^2\theta+a^2r^2\sin^2\theta=b^4-2b^2c\cos\theta. \tag{15}$$

In order to simplify further, we can change (15) to include only cosine by using the trigonometric identity ($\sin^2\theta=1-\cos^2\theta$).

$$b^2r^2\cos^2\theta+a^2r^2-a^2r^2\cos^2\theta=b^4-2b^2c\cos\theta$$

$$\left(r^2\cos^2\theta\right)\left(b^2-a^2\right)+a^2r^2=b^4-2b^2c\cos\theta$$

$$c^2r^2\cos^2\theta-2b^2c\cos\theta+b^4=a^2r^2$$

$$\left(cr\cos\theta-b^2\right)^2=\left(ar\right)^2$$

$$cr\cos\theta-b^2=-ar$$

Note in this last step, the negative root of *ar* was taken. If the positive root was taken, the final answer would be negative for all values of θ, but r, the distance from the sun to a planet, can only be positive. This leads us to

$$r=\frac{b^2}{a+c\cos\theta}.$$

The following are standard equations that relate the different measurements of an ellipse to each other. "e" denotes the eccentricity of an ellipse, which in rough terms measures how close the ellipse is to a circle. Values of e range from 0 to 1, or perfectly circular to highly elongated.

$$c=ae; b^2=a^2-c^2\rightarrow b^2=a^2-a^2e^2$$

$$r=\frac{a^2-a^2e^2}{a+ae\cos\theta}$$

$$r=\frac{a\left(1-e^2\right)}{1+e\cos\theta} \tag{16}$$

Simplifying Irregularities in Elliptical Motion

After determining an equation that defines planet location along their orbit, we have to tackle the problem of motion along this path. When a planet revolves around the sun, its speed is always different depending on its distance from the sun. Rather than trying to find an equation that will determine the velocity of the planets at a given time, it is easier to relate elliptical motion to some value that changes at a constant and definable rate. For this process, our ellipse is moved back to its original position, centered on the origin, with a circle drawn around it with radius, a (Fig. 6).

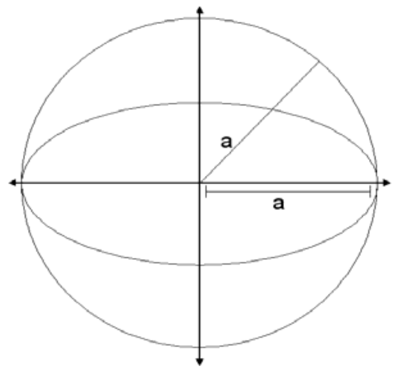



Fig. 6: An auxiliary circle with radius a.

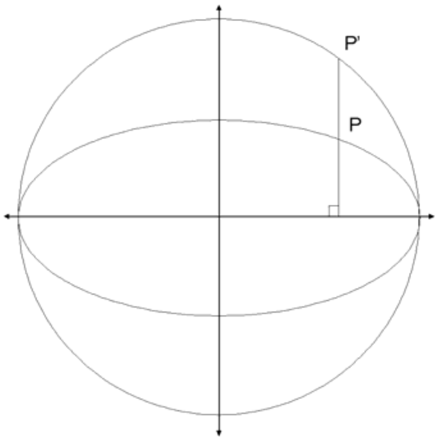



Fig. 7: On the auxiliary circle, there exists point P'.

From any point on the ellipse, P, there exists a point, P', on the auxiliary circle that lies on the shortest line perpendicular to the x-axis through point P (Fig. 7).

The angle created by P' and the x-axis through the origin is termed E. The angle created by P and the x-axis through the focus is still θ. While θ changes at an irregular rate, E changes much more regularly. Thus, we can establish a relationship between θ and E which will eventually be used to define the motion of the planets as a constant:

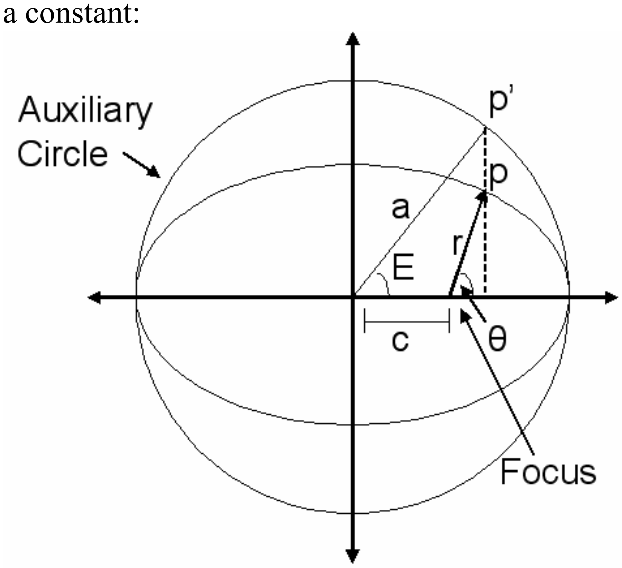



Fig. 8: An ellipse with constituent parts a, E, c, $\theta$, point P, and point P'.

$$a\cos E = x$$

$$r\cos\theta = x - c$$

$$a\cos E - ae = r\cos\theta$$

Remember now that by shifting the ellipse and converting its formula to polar coordinates, we established a relationship (17) between r and θ:

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\theta} \tag{17}$$

Thus,

$$\cos E = \frac{e+\cos\theta}{1+e\cos\theta} \tag{18}$$

Unifying Elliptical Geometry with Kepler's Laws

Because $M = E - e\sin E$ changes uniformly, it is convenient to find an equation that relates $\frac{dM}{dt}$ to other constants. Since $\frac{dM}{dt}$ is the rate, $\frac{dM}{dt}$ must be a constant.

Differentiate

$$M = E - e\sin E \tag{19}$$

To get

$$\frac{dM}{dt} = \frac{dE}{dt}\left(1 - e\cos E\right). \tag{20}$$

Further differentiate

$$\cos E = \frac{e + \cos\theta}{1 + e\cos\theta} \tag{21}$$

To get

$$-\sin E\frac{dE}{dt} = \frac{\left[(-\sin\theta)(1 + e\cos\theta) - (-e\sin\theta)(e + \cos\theta)\right]}{(1 + e\cos\theta)^2}.$$

Substitute (21) into (20) to get

$$\frac{dM}{dt} = \frac{dE}{dt}[1 - e(\frac{e + \cos\theta}{1 + e\cos\theta})].$$

Distributing e, we get

$$\frac{dM}{dt} = \frac{dE}{dt}\left[\frac{\left(1 - e^2\right)}{1 + e\cos\theta}\right]. \tag{22}$$

We know from (17) that

$$r = a\frac{\left(1 - e^2\right)}{1 + e\cos\theta}.$$

Substitute (17) into (22) to get:

$$\frac{dM}{dt} = \frac{dE}{dt}\cdot\frac{r}{a},$$

$$\frac{dM}{dt} = \frac{\sin\theta}{1 + e\cos\theta}\cdot\frac{1}{-a\sin E}\cdot\frac{h}{r}\cdot\frac{r}{a},$$

$$\frac{dM}{dt} = \frac{\sin\theta}{1+e\cos\theta} \cdot \frac{1}{-\sin E} \cdot \frac{h}{a^2}. \tag{23}$$

Using trigonometric substitution on equation (21), we know that

$$\pm \sin E = \sqrt{1 - \left(\frac{e+\cos\theta}{1+e\cos\theta}\right)^2}.$$

$$\pm \sin\theta = \sqrt{1-\cos^2\theta}$$

It is possible to insert these equations into the (23) and get

$$\frac{dM}{dt} = \frac{\sqrt{1-\cos^2\theta}}{1+e\cdot\cos\theta} \cdot \frac{1}{\sqrt{1+\frac{e^2 2e\cos\theta + \cos^2\theta - \cos^2\theta}{(1+e\cos\theta)^2}}} \cdot \frac{h}{a^2}.$$

Through further expansion, we get

$$\frac{dM}{dt} = \frac{\sqrt{1-\cos^2\theta}}{1+e\cdot\cos\theta} \cdot \frac{\sqrt{(1+e\cos\theta)^2}}{\sqrt{1+2e\cos\theta+\cos^2\theta - e^2 - 2e\cos\theta - \cos^2\theta}} \cdot \frac{h}{a^2}$$

Recall (12):

$$h = \sqrt{1-e^2} * \sqrt{GMa}$$

Simplify using this equation and get

$$\frac{dM}{dt} = \frac{1}{\sqrt{1-e^2}} * \frac{h}{a^2}$$

$$\frac{dM}{dt} = \frac{\sqrt{GMa}}{a^2}$$

Previously, we had the equation (13):

$$T^2 = \frac{4\pi^2}{GM} * a^3$$

With this equation (13), solving for *T*, we get

$$T = \frac{2\pi a\sqrt{a}}{\sqrt{GM}},$$

$$\sqrt{GM} = \frac{2\pi a\sqrt{a}}{T}.$$

Plugging this into the main equation (27), we get

$$\frac{dM}{dt} = \frac{2\pi a}{Ta} = \frac{2\pi}{T}$$

So the final result is that

$$\frac{dM}{dt} = \frac{2\pi}{T}.$$

Integrating, and finding the constant of integration to equal zero, we conclude:

$$M = \frac{2\pi}{T}t$$

To demonstrate a usage of the derived equations, the following problem was drawn up: given a date of the year, find the reference angle θ of the Earth in relation to its position when it is the closest to the sun. In this case, we chose the date August 4, 2005. The point at which the Earth is closest to the sun, the perihelion, is found at t = 0, and is around January 4 each year. August 4 is 212 days after January 4; thus for this situation, t = 212.

Then, plugging in the values T = 365.25 (the period of the Earth) and t = 212, we get

$$M \approx 3.6469 \text{ radians.}$$

Because M = E – e sin E (22),

$$E - e \sin E \approx 3.6469 \text{ radians.}$$

The value of the constant e, the eccentricity of the orbit of the Earth, is 0.0167.

$$E - (0.0167) \sin E \approx 3.6469 \text{ radians}$$

Using a graphing calculator, or using mathematical manipulation, we find that

$$E \approx 3.6389 \text{ radians.}$$

Using the previously determined equation (21) $\cos E = \frac{e + \cos\theta}{1 + e\cos\theta}$, we find cos E = cos (3.6389… radians), and

$$-0.8788 \approx \frac{e + \cos\theta}{1 + e\cos\theta}.$$

Let α = -0.8788. We find

$$\alpha = \frac{e + \cos\theta}{1 + e\cos\theta}.$$

Using algebraic manipulation, we get

$$\cos\theta = \frac{\alpha - e}{1 - \alpha * e}.$$

Substituting back in α and e, we get

$$\cos\theta \approx -0.8826,$$

$$\theta \approx 2.6521 \text{ radians} \approx 208.0430 \text{ degrees}.$$

Thus, the angle of the Earth in relation to its point at its perihelion is approximately 28.0430 degrees past 180 degrees, or about 208.0430 degrees.

**PART III: SPHERICAL TRIGONOMETRY**

To determine the location of the planets in reference to the night sky, it is necessary to derive functions that work for spherical triangles, since we assume for simplicity that the planets travel on a celestial sphere.

Law of Cosines

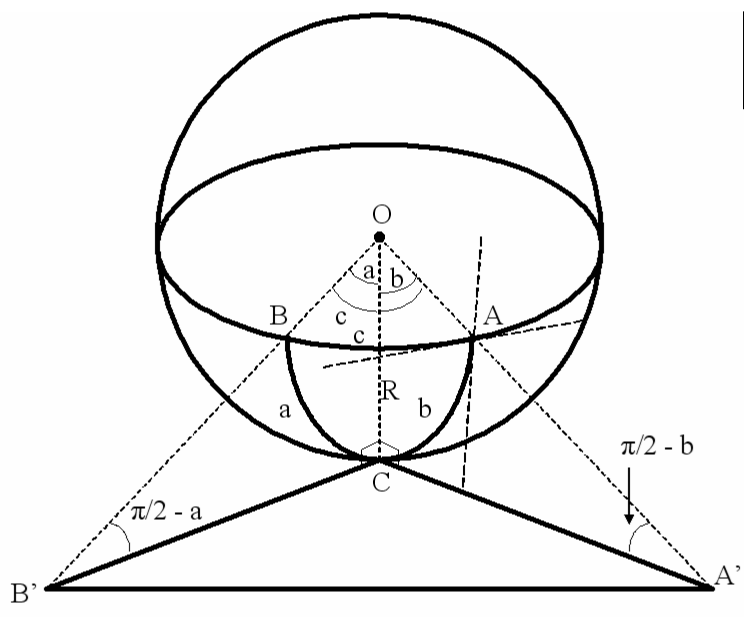



Fig. 9: The points of a spherical triangle projected on a plane.

Three intersecting arcs of distinct great circles form a spherical triangle. Sides are given by angles rather than lengths. Each angle of a spherical triangle is defined as the angle between the lines tangent to the arc at the vertex of the spherical triangle (Fig. 9). To translate these shapes to a Euclidean system, we project the spherical triangles to a plane. We start with triangle ABC on a sphere of radius R (Fig. 9). To project the spherical triangle, we place the sphere tangent to a plane at C.

We draw a radius from point O to point C. The radii through points A and B are then extended onto the plane on which the sphere rests to points A' and B', respectively. Points A', B', and C form a projected triangle on the plane. The planar triangle is the base of tetrahedron A'B'CO (Fig. 3.2). Since B'C and A'C are the tangent lines to arcs b and a, respectively, the angle BCA on the spherical triangle is the same as angle B'CA' on the planar triangle. For face B'OC, we apply the Law of Sines,

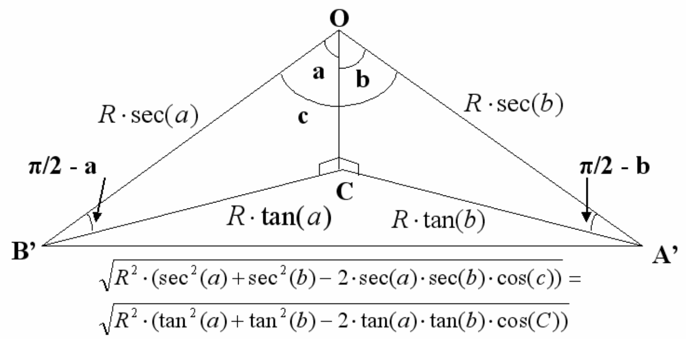



Fig. 10: The law of cosines is applied to the new polar variables.

$$\frac{R}{\sin(\pi/2-a)} = \frac{OB'}{\sin(\pi/2)} \Rightarrow R\sec(a) = OB'$$

From this, we get

$$\tan(a) = \frac{B'C}{R} \Rightarrow B'C = R\tan(a) .$$

We then carry out an analogous process on triangle A'OC and get OA' = $R$ sec(b) and A'C = $R$ tan(b). To find side B'A', we use the Law of Cosines first on triangle B'CA', and then on triangle B'OA':

$$(B'A')^2 = (R\tan(a))^2 + (R\tan(b))^2 - 2\cdot(R\tan(a))\cdot(R\tan(b))\cdot(\cos(C))$$
$$(B'A')^2 = (R\sec(a))^2 + (R\sec(b))^2 - 2\cdot(R\sec(a))\cdot(R\sec(b))\cdot(\cos(c))$$

This implies:

$$\sec^2(a) + \sec^2(b) - 2\cdot\sec(a)\cdot\sec(b)\cdot\cos(c) = \tan^2(a) + \tan^2(b) - 2\cdot\tan(a)\cdot\tan(b)\cdot\cos(C)$$

Using algebra, we get

$$2 - 2\cdot\sec(a)\cdot\sec(b)\cdot\cos(c) = \tan(a)\cdot\tan(b)\cdot\cos(c)$$

And finally

$$\cos(c) = \cos(a)\cdot\cos(b) + \sin(a)\cdot\sin(b)\cdot\cos(C) . \qquad (24)$$

Law of Sines

Starting with the Law of Cosines,

$$\cos(c) = \cos(a)\cdot\cos(b) + \sin(a)\cdot\sin(b)\cdot\cos(C) ,$$

We isolate the angle that we are looking for (C)

$$\cos(C) = \frac{\cos(c) - \cos(a)\cdot\cos(b)}{\sin(a)\cdot\sin(b)} .$$

We then substitute in for cosine:

$$\sqrt{1-\sin^2(C)} = \frac{\cos(c)-\cos(a)\cdot\cos(b)}{\sin(a)\cdot\sin(b)}.$$

Then we apply some algebra and get

$$\sin^2(C) = \frac{\sin^2(a)\cdot\sin^2(b)-\cos^2(c)+2\cdot\cos(a)\cdot\cos(b)\cdot\cos(c)-\cos^2(a)\cdot\cos^2(b)}{\sin^2(a)\cdot\sin^2(b)}.$$

Substituting (1-$\sin^2$(x)) for all $\cos^2$(x) we get

$$\sin^2(C)\cdot\sin^2(a)\cdot\sin^2(b) = \sin^2(c)+\sin^2(a)+\sin^2(b)+2\cdot\cos(a)\cdot\cos(b)\cdot\cos(c).$$

We recognize that the right side of the equation remains constant regardless of whether we perform this process for angle A, B, or C. Therefore, we equate the following and get

$$\sin^2(C)\cdot\sin^2(a)\cdot\sin^2(b) = \sin^2(B)\cdot\sin^2(a)\cdot\sin^2(c) = \sin^2(A)\cdot\sin^2(c)\cdot\sin^2(b).$$

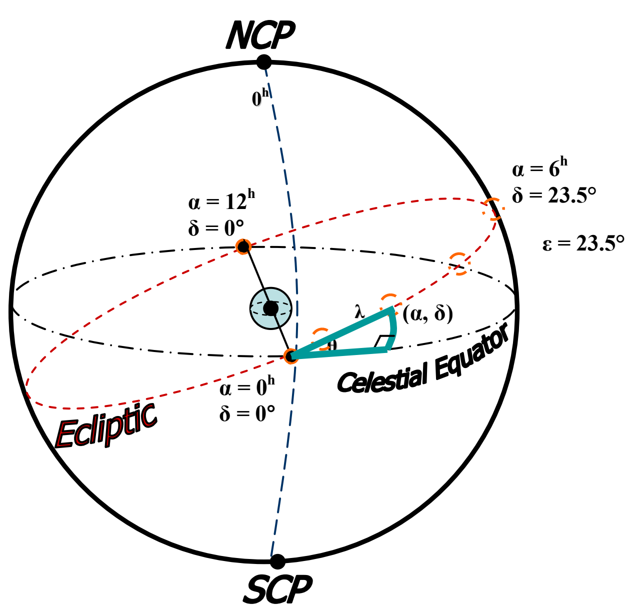
From this we obtain the following as a Law of Sines for spherical triangles:

$$\frac{\sin(C)}{\sin(c)} = \frac{\sin(B)}{\sin(b)} = \frac{\sin(A)}{\sin(a)}.$$

Finding the position of objects in the celestial sphere

Using the Law of Sines and the Law of Cosines we derived for spherical trigonometry, we are able to find the position of a certain object, such as the sun, as seen from the Earth. In our model, we visualize a celestial sphere with the Earth at its center as the sun revolves around the planet (Fig. 3.3).

Fig 11: The sun is defined by (α, δ) on the celestial sphere.




The celestial sphere has defined northern and southern poles, as well as a celestial equator. This equator is a great circle, one whose diameter is the same as that of the sphere and is formed when a plane intersects the sphere through its center. The path on which the sun travels, the ecliptic, is also depicted in the figure. Objects seen on the celestial sphere from Earth are located on a coordinate system that utilizes right ascension (latitude, represented by α,) and declination (longitude, symbolized by δ.)

The range of α is from 0 to 24 hours, and δ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ radians. Using these units, one hour can be expressed as 15° (360°/24). We further define $\varepsilon$ to be the tilt of the earth's axis from the equator. A line connecting the North Celestial Pole and the South Celestial Pole is drawn on the celestial sphere, indicating where time equals 0 hours. This meridian is analogous to the imaginary line running through Greenwich, England on the globe. If the sun is positioned where this meridian intersects with the ecliptic, its right ascension and declination are both 0. At the time of the summer solstice, the sun has traveled one quarter of its path around the Earth and $\delta = \varepsilon = 23.5°$. The distance on the ecliptic from the "origin" ($\alpha = 0$, $\delta = 0$) over which the sun has traveled is labeled $\lambda$. As the sun journeys around the planet, its right ascension and declination will constantly be changing. Our objective was to find the position of the sun ($\alpha$, $\delta$) at any given time.

A ninety-degree angle is formed when the line where $\alpha = 0$ and $\delta = 0$ intersect with the celestial equator. If we apply the Law of Sines for spherical trigonometry to a triangle with one vertex at (0, 0), another on the celestial equator, and the last vertex indicating the position of the sun, then we get

$$\frac{\sin\lambda}{\sin 90°} = \frac{\sin\delta}{\sin\varepsilon}.$$

This statement can be simplified to yield the equation $\sin\delta = \sin\lambda \sin\varepsilon$. Finally, we see that

$$\delta = \text{Arcsin}(\sin\lambda \sin\varepsilon),$$

showing that the declination of the sun can be found using given values for $\lambda$ and $\varepsilon$.

The right ascension of the sun can also be obtained by applying the spherical Law of Cosines,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

to the same triangle used to find $\delta$. By substituting in variables, we find

$$\cos\lambda = \cos\alpha \cos\delta + \sin\alpha \sin\delta \cos\frac{\pi}{2},$$

$$\cos\lambda = \cos\alpha \cos\delta \;\Rightarrow\; \cos^2\lambda = \cos^2\alpha\,(1 - \sin^2\delta).$$

Using the earlier equation $\sin\delta = \sin\lambda \sin\varepsilon$, we derive

$$\cos^2\lambda = \cos^2\alpha\,(1 - \sin^2\lambda \sin^2\varepsilon),$$

$$\cos^2\alpha = \frac{\cos^2\lambda}{1 - \sin^2\lambda \sin^2\varepsilon}.$$

We finally determine that

$$\alpha = \arccos\left(\frac{\cos\lambda}{\sqrt{1 - \sin^2\lambda \sin^2\varepsilon}}\,\frac{12}{\pi}\right),$$

and we can conclude that the right ascension of the sun can also be obtained if $\lambda$ and $\varepsilon$ are given .

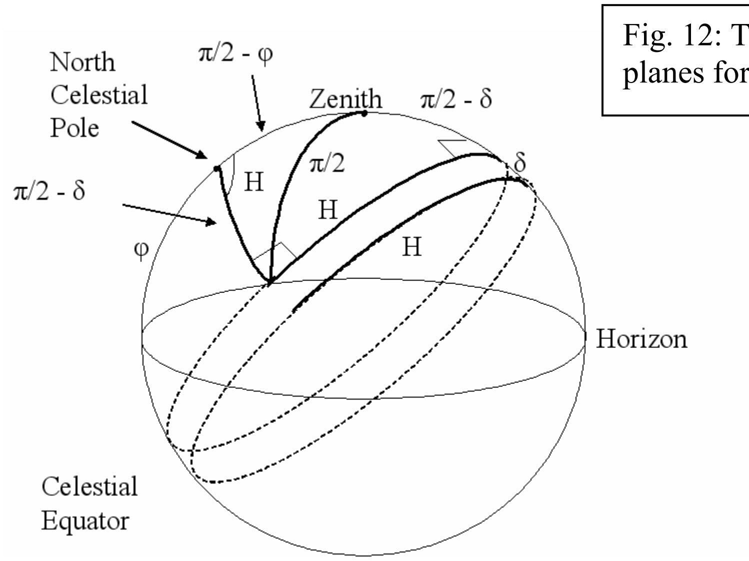



Fig. 12: The celestial sphere with circular planes for the horizon, equator, and sun.

Spherical triangles can be used to determine how long the sun is above the horizon at a given latitude. The horizon depends on the latitude of one's position on Earth. As the earth's radius is negligible when compared with the celestial sphere, we view the earth as a point. The horizon is defined as a plane passing through the point with its slope determined as relating angle $\varphi$, the latitude of the earth. This is represented on Fig. 12 as the angle between the line perpendicular to the horizon and the equator.

Define plane H as the plane passing horizontally through the celestial sphere. Crossing this great circle are two parallel circles on the celestial sphere. The equator is projected onto the celestial sphere and named plane E, while the other circle depicts the path of the sun on the celestial sphere at a given latitude, named plane S. The arc between these two parallel circles is defined as $\delta$. The angle and arc length between the equator circle and the horizon is $(\pi/2-\varphi)$. The point Z is defined as the intersection of the line perpendicular to the horizon emerging from the center of that circle and the celestial sphere. The north celestial pole (NCP) is similarly defined as the intersection of the line perpendicular to the equator circle emerging from the center of that circle and the celestial sphere.

From Fig. 3.4, we can find the amount of time the sun spends above the horizon in one day. This value can be calculated from the ratio of the circumference of great circle S above plane H to the total circumference. To accomplish this, first consider point A on plane H, where the sun's path crosses the horizon. Because the great circle containing Z is perpendicular to H at point A, arc AZ has the length $\pi/2$. Joining point N with A produces a side of length $(\pi/2-\delta)$ that connects arc $\delta$ with the point perpendicular to horizon. As the angle between the horizon and the NCP is $\varphi$, the angle between point N and Z is $(\pi/2-\varphi)$. Finally, we must determine the measure of angle H in Fig. 3.4, formed by the sides AN and NZ. First, we can connect point N and plane E with two arcs of length $\pi/2$. The first will pass through point A, while the second will pass through point Z. We can define point B as the intersection of the arc containing point Z and plane S.

Recalling the original question, consider the side lengths AB and DF, as well as angle H. Because DF lies on a great circle, and angle H is on the line perpendicular to the center of plane E, the side length of DF corresponds to angle H. Since both the side lengths and angle measures are in degrees, we recognize that the angle measure of DF will equal that of AB. Since AB represents only one-half of the sun's path over the horizon during one day, (AB/180)24 will equal the total number of hours the sun spends above ground in one full day. To find this value, we use the spherical trigonometric Law of Cosines (27), with angle H being equal to length AB. We get

$$Cos(\pi/2) = Cos(\pi/2-\delta)\cdot Cos(\pi/2-\varphi)+Sin(\pi/2-\delta)\cdot Sin(\pi/2-\varphi)\cdot Cos(AB)\,.$$

Using basic trigonometric identities, we determine that

$$0 = Sin(\delta)\cdot Sin(\varphi)+Cos(\delta)\cdot Cos(\varphi)\cdot Cos(AB)\,.$$

This provides the final answer, expressed in radians. We get

$$AB = Arc\cos(-Tan(\delta)\cdot Tan(\varphi))\,.$$

The number of hours spent by the sun above the horizon can therefore be defined as:

$$Hours = (2/15)\cdot Arc\cos(-Tan(\delta)\cdot Tan(\varphi))\,.$$

**PART IV: THE SUNDIAL**

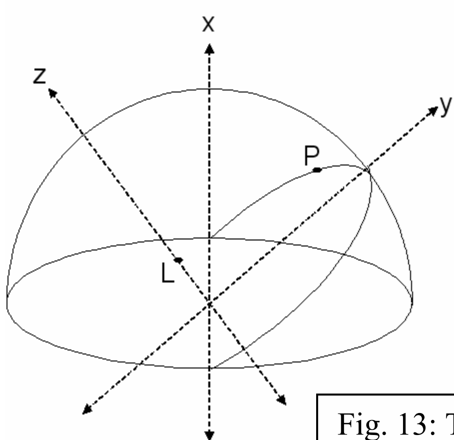



Fig. 13: The initial reference frame.

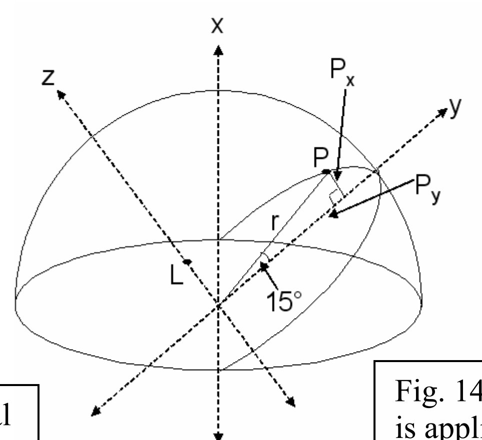
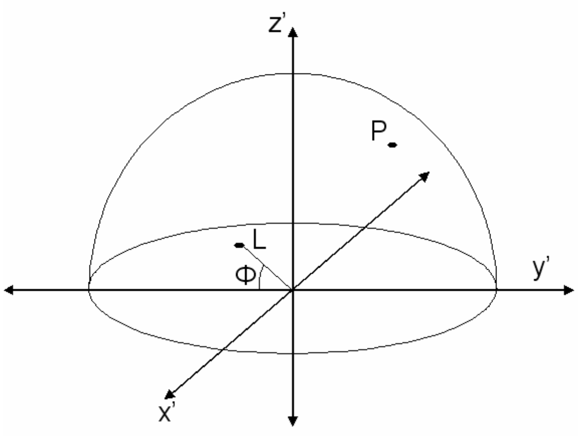



Fig. 14: Vector L is applied to the reference frame.

Since the focus of the project was to understand the complexities of celestial motion, a small group of us derived the necessary angles, and set out to make a sundial. Sundials use an object aligned at a specific angle to cast a shadow on the ground. The position of the shadow relative to the object casting the shadow is used to represent time. Fig. 13 is the reference frame used initially. This reference frame was chosen because the points used can be represented in one or two dimensions. The vector L describes the distance from the ground to the top of the object, and the point P represents the point where the sun is (Fig. 14).

Currently, we are in daylight savings time, which differs from "sun time" (or actual time) by one hour. Therefore, when the sun is at the 11:00 position in the sky, clocks in daylight savings time read 12:00. The point at which the sun is in the 11:00 position will be represented by the point P.

Fig. 15: The new coordinate system.




Assuming that the earth rotates uniformly, the sun should move uniformly across the sky relative to the earth. Since there are 24 hours in a day and 360 degrees in a circle, each hour, the sun should move 15 degrees along its path in the sky. We then proceeded to determine the location of the points chosen relative to the reference frame selected. Using simple trigonometric functions, we can now easily determine the positions of the points P and L.

L = <0, 0, L> (Fig. 14)
P = <-r*sin(15), r*cos(15), 0> (Fig. 14)

Now that we have these coordinates, rotate the frame of reference around the X axis by Φ degrees, the angle to rotate the Y and Z axes, such that the X' and Y' axes form a coordinate plane on the surface of the earth (Fig. 15) Thus, we can find new coordinates <X',Y',Z'> for L and P in relation to the earth's surface, with X = X'.

Considering X = X', now solve for the points P and L in two dimensions. Under the new coordinate system, P = <-rsin(15), rcos(15)sin(Φ), rcos(15)cos(Φ)>, and L = <0, -Lcos(Φ), Lsin(Φ)>. Using the standard form of a line for three dimensions, $\vec{r} = r + \vec{v}t$ with $r = L$ and $\vec{v} = (P - L)$, we get

$$\vec{r} = < o, -L\cos\phi, L\sin\phi > + < -r\sin 15, (r\cos 15\sin\phi + L\cos\phi), (r\cos 15\cos\phi - L\sin\phi) > t.$$

With that said, we now can solve for the original goal. Light from the sun will be emitted from point P and cross the tip of L, and hit the ground (Z'= 0), forming a shadow at an angle $\theta$ with respect to Y'. Since at 12:00, $\theta$=0, at other time points, $\theta$ is equal to the angle that the shadow makes with the 12:00 line. Using that data, it is possible to construct the clock. We solved for $\theta$.

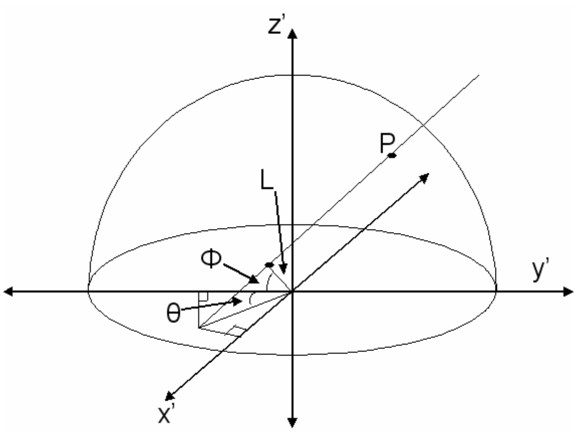



Fig. 16: Light hits the ground at point L.

Using the equation of the line formed between the points P and L, we can easily solve for the point at which the line hits the ground (Z'= 0). Once we know that point, we would be able to solve for $\theta$ using the tangent function. The general equation for the different components of a line in three dimensions is as follows:

$$x = x_0 + v_1 t$$
$$y = y_0 + v_2 t$$
$$z = z_0 + v_3 t$$

We know that since *Z'*=0, z =0, so we can solve for *t* when z =0 like so:

$$0 = L\sin\phi + (r\cos 15\cos\phi - L\sin\phi)t$$
$$t = \frac{-L\sin\phi}{r\cos 15\cos\phi - L\sin\phi}$$

Now that we have *t* when $z = 0$, we can get the other components of that point.

Since $\tan\theta = \frac{o}{a}$, $\tan\theta = \frac{X}{Y}$. With that information, we then solve for $\tan\theta$ as follows:

$$\tan\theta = \frac{\frac{-r\sin 15(-L\sin\phi)}{r\cos 15\cos\phi - L\sin\phi}}{-L\cos\phi + (r\cos 15\sin\phi + L\cos\phi)\frac{-L\sin\phi}{r\cos 15\cos\phi - L\sin\phi}}$$
$$\tan\theta = \frac{Lr\sin 15\sin\phi}{L\cos\phi(r\cos 15 - L\sin\phi) + Lr\cos 15\sin^2\phi + L^2\cos\phi\sin\phi}$$
$$\tan\theta = \frac{Lr\sin 15\sin\phi}{Lr\cos 15(\sin^2\phi + \cos^2\phi)}$$
$$\tan\theta = \tan 15\sin\phi \qquad (25)$$

Using that equation (28), we now obtain values of $\theta$ for positions of the sun other than 15 degrees. To figure out the other positions of the sun, we just substituted in 30 for 10:00/2:00, 45 for 9:00/3:00, and 60 for 8:00/4:00. As stated earlier, $\phi =$ the degree of latitude that the sundial is at. With all that information, we were ready to go out to the field and put our predictions to practicality.

<u>Building the Sundial</u>

We found a reference line (the 12:00 PM line for the clock) the following day using a long pole that we made perpendicular to the ideal earth's surface (a perfect sphere). The reference line is the line created by the shadow formed by a pole perpendicular to the earth at 12:00 sun time (1:00 PM). Using string and nails, we marked this line. We calculated the necessary angles using equation (28). Using the tan function and simple distance measurements along the reference line, we mapped out the angles we calculated and used nails and string to mark them. We used a board to hold the pole casting the shadow at 41 degrees, the approximate value for our degree of latitude. The reason the pole is at a slant is to account for our position on the earth in relation to the equator, such that the sun's rays hit the pole perpendicularly. We then checked our sundial against the sun and found that we had the correct time, and spray painted the final product onto the ground, removing the strings of the original dial.

**CONCLUSION**

Studying the laws of celestial mechanics requires the integration of physics, geometry, trigonometry, calculus, and astronomy. Using our collective knowledge of these related fields we derived the original equations of Johannes Kepler in order to better understand planetary motion. The information required divided our team into three primary areas; the first group tackled Kepler's primary works and succeeded in deriving his three laws, the second group explored the patterns and equations associated with the geometry of ellipses and simplified them for more practical applications, and the third group transferred their knowledge of triangles to spherical trigonometry which more accurately depicts the night sky as viewed from Earth. Then coming together, we provided a more coherent view of celestial mechanics. The study of ellipses created equations that, with Kepler's Laws, allowed the group to characterize any planet's motion as a constant. This could be applied to find a planet's position with respect to the sun. The spherical trigonometry group could then take this information and place each planet in the night sky for our observation. Our team then derived the equations and measurements associated with a sundial by studying how the sun moves across the sky and then constructing a working model.

Our next step would have been to determine the locations of the planets, both along their orbit and in the sky. We could have then tested these results through the Drew University observatory. Today, the discoveries that we made in team project are used by astronomers everywhere to predict the motions of the heavenly bodies. In a recent mission, NASA scientists were able to accurately predict the motion of a distant comet and hit it with an impact probe launched from Earth, no doubt using some form of Kepler's Laws and the related equations that we discovered in the classroom. Thus, the practical importance of our work in this team project is unquestionable and will surely continue to serve society as we to look to the sky with wonder.

Aside from the wider affects of this work, the fact remains that the project as a whole went quite smoothly. The fact that it was rarely necessary to ask for help in the derivation of some of the most important formulas in basic astronomy is a credit to the talent and hard work of this team. Regardless of the age of these formulas, this project on the whole was an overwhelming success.