CELESTIAL MECHANICS

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ABSTRACT

The goal of this project was to explore the mechanics of the heavenly bodies – the Sun, the planets, and the stars. To do this, Kepler's Laws of Planetary Motion were derived through the use of Newton's Laws of motion, and analysis of elliptical geometry yielded mathematical solutions to the orbital paths of the planets. Spherical triangles were used to analyze the celestial sphere and to ultimately analyze the motion of the other heavenly bodies in this sphere with respect to a stationary observer. Ultimately, a model was created that could determine the location of a planet on any given day, and a further geometrical analysis allowed for the construction of a working sundial, which could tell time to a reasonable degree of accuracy. The work done on the project provided a practical application of Kepler's Laws and both planar and spherical geometry, and led to a better understanding of the workings of motion as related to the celestial sphere.

INTRODUCTION

Since the earliest civilizations, humans have been intrigued by the strange workings of the heavens above. Over the millennia, the planets have been used to predict the weather, to determine one's personality, and even to represent the gods. The great power that was ascribed to the planets seemed to be a representation of the mystery they held for us, for nobody truly understood just how these bodies moved about the sky. In an effort to explain the movements of the heavens, such thinkers as Aristotle, Ptolemy, and Copernicus introduced their theories. Though these theories gradually improved our understanding of the universe, their predictions were not completely consistent with observation. This state of affairs was revolutionized by Johannes Kepler in the early 1600s. After studying the observations of astronomer Tycho Brahe, Kepler introduced his three laws of planetary motion, which for the first time accurately explained the motions of celestial bodies. In this project we explored Kepler's Laws and their applications in locating celestial bodies in the sky.

PART I: THE GEOMETRY OF THE ELLIPSE

As planets orbit the sun, their paths follow elliptical curves. Planetary motion is closely intertwined with the geometry of the ellipse. As such, we want to study the equations associated with the ellipse to gain a better understanding of the curve. This will eventually help us to accomplish the task of tracking planetary motion.

The Polar Form of the Ellipse

We want to be able to find the distance from the Earth to the sun at any time. Therefore, we want to find the equation of an ellipse as a function of polar coordinates. As we will show later, for all planets, the sun lies at a focus on an elliptical orbit. Therefore, we are first interested in translating the ellipse such that a focus (and thus the sun) lies at the origin of a Cartesian plane. The formula for an ellipse centered at point (*h*,*k*) in rectangular coordinates is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

where a is the length of the semi-major axis, b is the length of the semi-minor axis, and $a > b$. An ellipse with a focus at the origin is

$$\frac{(x+c)^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

where c is the distance from the origin to a focus of the ellipse. This models a planetary orbit with the sun at the origin. If we convert this equation to a function of polar coordinates, we will obtain a function that relates r, a planet's distance from the sun, with θ, the angle that the planet makes with the sun and the x-axis. We use polar coordinate definitions to convert (1) to polar form.

$$\frac{(r\cos\theta+c)^2}{a^2}+\frac{(r\sin\theta)^2}{b^2}=1 \tag{2}$$

In any ellipse, it is given that

$$c=ea \text{ and } b^2=a^2-c^2 \tag{3}$$

where e is defined as the eccentricity of the ellipse. Eccentricity is a measure of how much a conic section deviates from a circle. Values of e range from 0 to 1; when $e = 0$, the ellipse is a perfect circle and when $e = 1$, the ellipse is a line segment. For Earth, $e = 0.0167$ [1]. Using (3) to rewrite (2) yields

$$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)+r\left(\frac{2e\cos\theta}{a}\right)+(e^2-1)=0$$

We can now solve for r by using the quadratic formula:

$$r=\frac{-\left(\frac{e\cos\theta}{a}\right)\pm\left(\frac{1}{a}\right)}{\left(\frac{\cos^2\theta}{a^2}\right)+\left(\frac{\sin^2\theta}{b^2}\right)} \tag{4}$$

Manipulating the equations from (3) gives

$$\frac{a^2}{b^2}=\frac{a^2}{a^2-c^2}=\frac{1}{1-\left(\frac{c^2}{a^2}\right)}=\frac{1}{1-e^2} \tag{5}$$

We simplify the complex fraction in (4), substituting in the equation from (5). Additionally, in polar form, r represents a distance from the origin, so we may eliminate the negative solution.

$$r=\frac{a(1-e^2)}{1+e\cos\theta} \tag{6}$$

Therefore, (6) is the equation of an ellipse with a focus at the origin written as a function of polar coordinates.

PART II: KEPLER'S LAWS

We will now prove Kepler's Laws, which serve as the basis for celestial mechanics because they describe the orbits of the planets. When Kepler published his three laws of planetary motion, his research was based off of careful and arduous observations. The mathematics and physics required to prove his laws were not available to Kepler, and a formal proof of his laws would not appear until Isaac Newton dictated his laws of motion and invented what we now know as calculus. Here, we will provide a proof for Kepler's revolutionary ideas.

Kepler's three laws can be obtained from Newton's laws and the gravitational force equation:

$$\mathbf{F} = m\mathbf{a}$$

$$\mathbf{F}_{\mathbf{G}} = \frac{-GMm}{r^2}$$

Kepler's First Law

We begin by proving Kepler's First Law, which states that the orbit of a planet can be described as an ellipse with the Sun at a focus. Therefore, we want to show that the orbit of a planet can be written as the polar equation in (6).

For the basis of our derivation, we assume that Newton's Second Law of Motion and Law of Gravitation are true. Combining the two laws, we obtain:

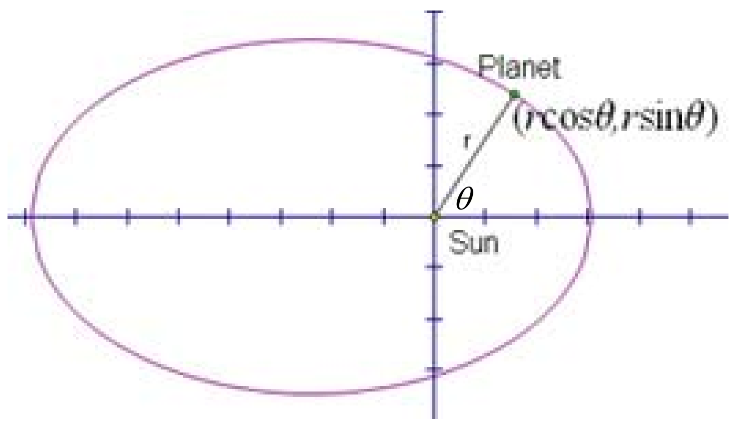

Fig. 1: Position of a planet with respect to the Sun in rectangular coordinates.

$$\mathbf{F}_{\mathbf{x}} = m\mathbf{a}_{\mathbf{x}} = \frac{-GMm}{r^2}\cos\theta \tag{7}$$

$$\mathbf{F}_{\mathbf{y}} = m\mathbf{a}_{\mathbf{y}} = \frac{-GMm}{r^2}\sin\theta \tag{8}$$

Note that acceleration is written as the second derivative of position with respect to time. From Figure 1, we see that the position of planet can be written in polar form where $x = r\cos\theta$ and $y = \sin\theta$. The second derivatives of x and y can be used to replace a_x and a_y.

$$\frac{d^2x}{dt^2} = -r[(\frac{d\theta}{dt})^2\cos\theta + \frac{d^2\theta}{dt^2}\sin\theta] - 2\frac{dr}{dt}\frac{d\theta}{dt}\sin\theta + \frac{d^2r}{dt^2}\cos\theta \tag{9}$$

$$\frac{d^2y}{dt^2} = -r[(\frac{d\theta}{dt})^2\sin\theta - \frac{d^2\theta}{dt^2}\cos\theta] + 2\frac{dr}{dt}\frac{d\theta}{dt}\cos\theta + \frac{d^2r}{dt^2}\sin\theta \tag{10}$$

We then substitute the second derivatives (9) and (10) into (7) and (8), respectively

$$\frac{-GM}{r^2}\cos\theta = -r[(\frac{d\theta}{dt})^2\cos\theta + \frac{d^2\theta}{dt^2}\sin\theta] - 2\frac{dr}{dt}\frac{d\theta}{dt}\sin\theta + \frac{d^2r}{dt^2}\cos\theta \tag{11}$$

$$\frac{-GM}{r^2}\sin\theta = -r[(\frac{d\theta}{dt})^2\sin\theta - \frac{d^2\theta}{dt^2}\cos\theta] + 2\frac{dr}{dt}\frac{d\theta}{dt}\cos\theta + \frac{d^2r}{dt^2}\sin\theta \tag{12}$$

To simplify, we multiply (11) by sin θ and (12) by cos θ. Subtracting the resulting equations yields:

$$0 = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \tag{13}$$

Multiplying (11) by sin θ and (12) by cos θ and adding the resulting equations yields:

$$\frac{-GM}{r^2} = -r(\frac{d\theta}{dt})^2 + \frac{d^2r}{dt^2} \tag{14}$$

Next, we integrate (13) and call the constant of integration h.

$$h = r^2\frac{d\theta}{dt} \tag{15}$$

We substitute this constant into (14).

$$\frac{-GM}{r^2} = \frac{-h^2}{r^3} + \frac{d^2r}{dt^2} \tag{16}$$

We want to determine a value of $\frac{d^2r}{dt^2}$ to substitute into (16). Define some u such that $r = \frac{1}{u}$.

The chain rule is used to calculate $\frac{d^2r}{dt^2}$

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{h}{r^2} = -h\frac{du}{d\theta}$$

$$\frac{d^2r}{dt^2} = -h\frac{d}{dt}[\frac{d}{d\theta}(\frac{du}{d\theta}\frac{d\theta}{dt})] = -h^2u^2\frac{d^2u}{d\theta^2} \tag{17}$$

Substituting (17) into (16) allows gravitational acceleration to be expressed in terms of u and θ

$$\frac{-GM}{h^2} = u + \frac{d^2u}{d\theta^2} \tag{18}$$

We then integrate (18) and solve for r, which yields

$$r = \frac{\frac{h^2}{GM}}{1 + \frac{h^2}{GM}(A\cos\theta + B\sin\theta)} \tag{19}$$

where A and B are constants of integration. We want r to be at a minimum when the planet lies on the positive x-axis. Thus, we want r to be at its minimum when $\theta = 0$. This implies that $B = 0$.

We use $e = \frac{Ah^2}{GM}$ and substitute this value into (19). This e is the eccentricity of a planet's elliptical orbit.

$$r = \frac{\left(\frac{e}{A}\right)}{1 + e\cos\theta} \tag{20}$$

The cosine function in the denominator in (20) translates into a periodic function, with a maximum and a minimum. We will determine the sum of the maximum and the minimum values of r and let it equal $2a$.

$$r_{\min} + r_{\max} = \frac{\left(\frac{e}{A}\right)}{1 + e\cos\theta} + \frac{\left(\frac{e}{A}\right)}{1 - e\cos\theta} = 2a$$

$$\frac{e}{A} = a(1 - e^2) \tag{21}$$

Substituting (21) into (19) yields

$$r = \frac{a(1-e)^2}{1 + ecos\theta}$$

Note that this is the same as the polar equation of an ellipse that we found in (6).

<u>Kepler's Second Law</u>

Kepler's Second Law states that the area of a sector swept out by a line segment from the Sun is equal for an equal time interval. To begin our proof, we define some variable A to be the area of a sector of an ellipse swept out an equal time interval, t. We want to prove that this area changes at a constant rate; that is, that $\frac{dA}{dt}$ is a constant. The area of any given sector of an ellipse in polar coordinates can be calculated as $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$. Therefore,

$$dA = \frac{1}{2} r^2 d\theta \tag{22}$$

To obtain $\frac{dA}{dt}$, we use the chain rule, substituting in (22) and (15).

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{h}{2} \tag{23}$$

As stated in (15), h is a constant, and therefore $\frac{h}{2}$ and $\frac{dA}{dt}$ are constant. The planets sweep out equal areas in equal amounts of time, thus proving Kepler's Second Law.

<u>Kepler's Third Law</u>

Kepler's Third Law states that the square of the period of a planet's orbit is proportional to the cube of its semi-major axis. The objective of this proof is to show that $\frac{T^2}{a^3}$ is a constant. We first integrate (19) with respect to t to obtain the area of the ellipse.

$$A = \int_0^T \frac{h}{2} dt = \frac{hT}{2} = \pi ab \tag{24}$$

Area of an ellipse in the Cartesian plane can also be calculated with the equation $A = \pi ab$, where a is the semi-major axis and b is the semi-minor axis. Also true for ellipses are $c = \sqrt{a^2 - b^2}$ and

$e = \frac{c}{a}$. Using the latter two properties of ellipses, we find that $b = a\sqrt{1-e^2}$. We substitute b into (24).

$$\frac{hT}{2} = \pi a^2 \sqrt{1-e^2} \tag{25}$$

We use (21) and $e = \frac{Ah^2}{GM}$ to solve for

$$h = \sqrt{GMa(1-e^2)} \tag{26}$$

We substitute this value of h into (25).

$$\frac{T\sqrt{GMa(1-e^2)}}{2} = \pi a^2 \sqrt{1-e^2}$$

Manipulating the latter result will lead to Kepler's Third Law.

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$

PART III: ELLIPTICAL MOTION

Relating an Ellipse to Its Circumscribed Circle

We have determined an equation defining a planet's orbit along an ellipse. Eventually, we would like to be able to find the angle θ at any time t. This, however, presents a problem, as θ does not change at a constant rate. The speed of a planet depends upon its distance from the sun. Therefore, we want to relate θ to some value that changes at a constant rate. To do this, we begin by translating an ellipse back to its original position, centered at the origin.

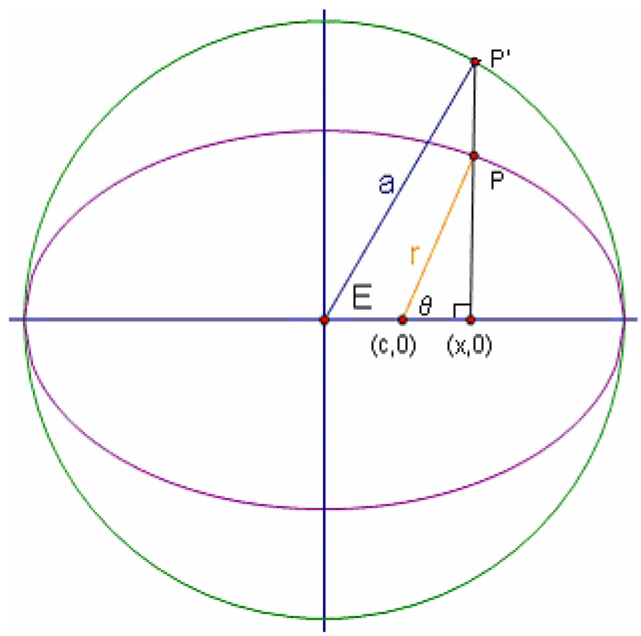

Fig. 2: Circle circumscribed about an ellipse.

We circumscribe a circle around the ellipse. For any point P on the ellipse, we define the corresponding point P' to be the point on the circle that lies on the line perpendicular to the x-axis through point P. Angle E is the angle in standard position such that E's terminal side is the ray connecting the origin to point P'. The distance from the origin to point P' is the circle's radius, which is also the ellipse's semi-major axis. Thus, the distance from the origin to point P' is a (Fig. 1). We want to find some function relating E and θ.

Through the use of trigonometry, we find that

$$a\cos E = x$$

$$r\cos\theta = x - c$$

$$a\cos E = r\cos\theta + ae \tag{27}$$

In Fig. 1, r and θ are related by the same function as in (6), because (6) was derived from a translated ellipse with a focus at the origin. Therefore, we can substitute r in (27) with our polar function from (6).

$$a\cos E = \frac{a(1-e^2)}{1+e\cos\theta}\cdot\cos\theta + ae$$

$$\cos E = \frac{\cos\theta + e}{1+e\cos\theta} \tag{28}$$

We know that for any angle α, the following trigonometric identity holds true:

$$\cos\alpha = \frac{1-\tan^2\left(\frac{\alpha}{2}\right)}{1+\tan^2\left(\frac{\alpha}{2}\right)}$$

Applying this identity to (28) gives us

$$\tan^2\left(\frac{E}{2}\right) = \frac{1-e}{1+e}\cdot\tan^2\left(\frac{\theta}{2}\right)$$

$$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\cdot\tan\left(\frac{\theta}{2}\right) \tag{29}$$

We now have obtained an equation relating θ and E. We are still unable find angle θ at any time t. However, E will play an important role in relating θ and t.

Kepler's Laws Applied to Elliptical Geometry

We still seek to find a way to relate the angle θ that a planet makes with its sun to time t. To do this, we define some quantity M such that $M = E - e\sin E$, where e is the eccentricity of a planet's orbit. We want to show that M's rate of change is constant. If we can show that M's rate of change is constant, we can calculate M at any time t, calculate E for any M, and calculate θ for any E. Thus, if we show that M's rate of change is constant, we can find angle θ at any time t.

The rate of change of M is given by its derivative with respect to time:

$$\frac{dM}{dt} = \frac{dE}{dt}\cdot(1-e\cos E) \tag{30}$$

We need to equate $\frac{dE}{dt}$. We can do this by taking the derivative of (29) with respect to time.

$$\frac{dE}{dt} = \sqrt{\frac{1-e}{1+e}}\cdot\frac{\sec^2\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{E}{2}\right)}\cdot\frac{d\theta}{dt}$$

Substituting in $\frac{d\theta}{dt}$ from (15) yields

$$\frac{dE}{dt}=\sqrt{\frac{1-e}{1+e}}\cdot\frac{\sec^2(\frac{\theta}{2})}{\sec^2(\frac{E}{2})}\cdot\frac{h}{r^2}$$

Substituting in the value of *r* from our polar function (6) yields

$$\frac{dE}{dt}=h\cdot\sqrt{\frac{1-e}{1+e}}\cdot\frac{\sec^2(\frac{\theta}{2})}{\sec^2(\frac{E}{2})}\cdot\frac{(1+e\cos\theta)^2}{a^2(1-e^2)^2} \tag{31}$$

We can then substitute (32) into (31), which gives us

$$\frac{dM}{dt}=h\cdot\sqrt{\frac{1-e}{1+e}}\cdot\frac{\sec^2(\frac{\theta}{2})}{\sec^2(\frac{E}{2})}\cdot\frac{(1+e\cos\theta)^2}{a^2(1-e^2)^2}\cdot(1-e\cos E) \tag{32}$$

We know that for any angle β, the following trigonometric identity holds true:

$$\cos(\frac{\beta}{2})=\pm\sqrt{\frac{1+\cos\beta}{2}}$$

Manipulating this identity yields

$$\sec^2(\frac{\beta}{2})=\frac{2}{1+\cos\beta} \tag{33}$$

We substitute (34) into (33)

$$\frac{dM}{dt}=h\cdot\sqrt{\frac{1-e}{1+e}}\cdot\frac{1+\cos E}{1+\cos\theta}\cdot\frac{(1+e\cos\theta)^2}{a^2(1-e^2)^2}\cdot(1-e\cos E) \tag{34}$$

We can use (28) to substitute for cos E in (35) to obtain

$$\frac{dM}{dt}=h\cdot\sqrt{\frac{1-e}{1+e}}\cdot\frac{1+\frac{\cos\theta+e}{1+e\cos\theta}}{1+\cos\theta}\cdot\frac{(1+e\cos\theta)^2}{a^2(1-e^2)^2}\cdot(1-e\cdot\frac{\cos\theta+e}{1+e\cos\theta})$$

$$\frac{dM}{dt}=h\cdot\sqrt{\frac{1-e}{1+e}}\cdot\frac{1+e\cos\theta+\cos\theta+e}{1+\cos\theta}\cdot\frac{(1+e\cos\theta)^2}{a^2(1-e^2)^2}\cdot(\frac{1-e^2}{1+e\cos\theta})$$

$$\frac{dM}{dt}=h\cdot\sqrt{\frac{1-e}{1+e}}\cdot\frac{1}{a^2(1-e)} \tag{35}$$

Substituting the value of *h* from (26) into (36) yields

$$\frac{dM}{dt}=\sqrt{aGM(1-e^2)}\cdot\sqrt{\frac{1-e}{1+e}}\cdot\frac{1}{a^2(1-e)}$$

$$\frac{dM}{dt}=\frac{\sqrt{GM}}{a^{\frac{3}{2}}} \tag{36}$$

Kepler's Third Law of planetary motion states that

$$\frac{T^2}{a^3}=\frac{4\pi^2}{GM}$$

$$a^{\frac{3}{2}} = \frac{T}{2\pi}\sqrt{GM} \tag{37}$$

Substituting (38) into (37) yields

$$\frac{dM}{dt} = \frac{2\pi}{T}$$

Recall that T is a constant representing the period of a planet's orbit. Therefore, $\frac{dM}{dt}$ is a constant, which means that M's rate of change is constant. We can integrate $\frac{dM}{dt}$ to give us an equation that expresses M in terms of time t

$$\int \frac{dM}{dt} = \int \frac{2\pi}{T}$$

$$M = \frac{2\pi t}{T} + C$$

where C is the constant of integration. At time t=0, angle E=0. Therefore, at time t = 0, C = 0. We conclude that

$$M = \frac{2\pi t}{T} \tag{38}$$

where T is the period of a planet's orbit, and t is the amount of time since a planet's perihelion, the closest approach of a planet to its sun. Earth's perihelion in 2006 was on Jan. 3, 2006 15:30 GMT [2]. Earth's distance from the sun at perihelion was 91,405,436 miles [3]. For any planet for which we know the period of orbit, we can find M at any time t after perihelion. This means that at for any time t, we are able to calculate the angle θ and the distance from the sun, r.

PART IV: SPHERICAL TRIGONOMETRY

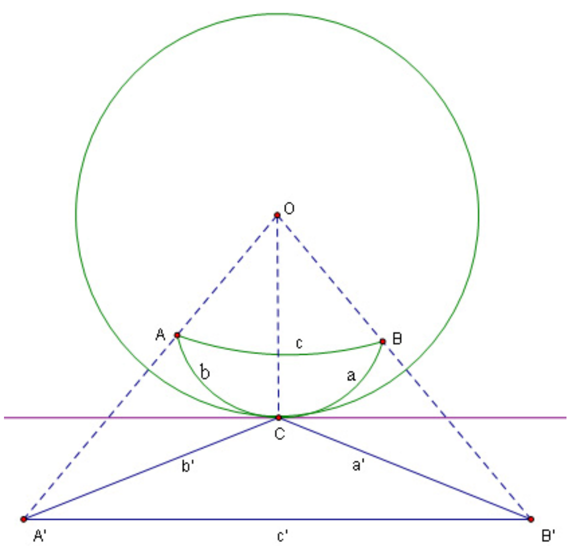

Fig. 3: Points A, B, C projected from center of sphere, O, onto the plane perpendicular to the sphere.

Spherical trigonometry studies the triangles formed by connecting three points on the surface of a sphere. The arcs that create the sides of the triangles are arcs that lie on the sphere's great circles, circles that have the same center as the given sphere.

Figure 3 is the construction of spherical triangle ABC on the surface of a sphere with center O. Points A, B, and C that form the spherical triangle $ABC1$ have sides a, b, and c. C on the sphere is also tangent to the plane. Three lines are extended from O through each point, A, B, C, until they intersect with the plane perpendicular to the sphere's bottom. These points A', B', and C are connected on the sphere to form a planar triangle $A'B'C$ with sides a', b', and c'.

The Spherical Law of Cosines

Starting with the given quantities r (radius); angles *A*, *B*, and *C*; and sides *a*, *b*, and *c*, we began to derive the spherical Law of Cosines from the planar Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

We began by finding pieces of the planar triangle via the given quantities in the spherical triangle.

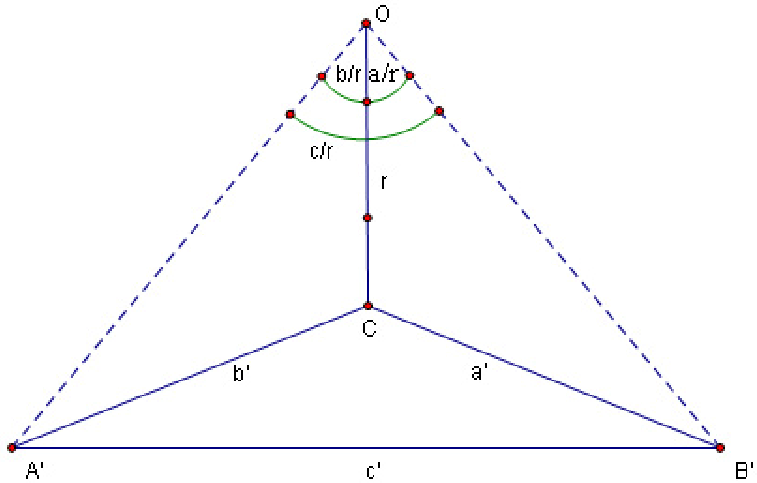

Fig. 4: Planar triangle A'B'C with each point extended to the origin to create an open triangular pyramid.

First we redefined the angles using the equation for an arc,

$$S = r\theta$$

where *S* is the arc length, *r* is the radius, and θ is the angle opposite the arc. From this we obtained angle *A'OC* in terms of *b* and *r* and angle *B'OC* in terms of *a* and *r*.

$$A'OC = \frac{b}{r}$$

$$B'OC = \frac{a}{r}$$

The tangent of right triangles *A'OC* and *B'OC* were utilized to define sides *a*' and *b*':

$$a' = r\tan\frac{a}{r}$$

$$b' = r\tan\frac{b}{r}$$

The law of cosines for *A'B'C* was written in terms of the givens by substituting the new values for *a'* and *b'* that were previously found.

$$c'^2 = a'^2 + b'^2 - 2a'b'\cos C$$

$$c'^2 = (r\tan\frac{a}{r})^2 + (r\tan\frac{b}{r})^2 - 2(r\tan\frac{a}{r})(r\tan\frac{b}{r})\cos C \qquad (40)$$

After finding the first equation for *c'*, we needed to derive a different equation for *c'* in order to equate them and find a Law of Cosines for a spherical triangle. We did this using the frontal triangle *A'OB'* in the open triangular pyramid. By using right triangles *A'OC* and *B'OC*, the Pythagorean theorem was applied to solve for sides *OB'* and *OA'* in terms of the given quantities.

$$(OA')^2 = r^2 + r^2\tan^2\left(\frac{b}{r}\right)$$

$$(OB')^2 = r^2 + r^2\tan^2\left(\frac{a}{r}\right)$$

These new values were plugged into the planar Law of Cosines:

$$c'^2 = r^2 + r^2 \tan^2\left(\frac{a}{r}\right) + r^2 + r^2 \tan^2\left(\frac{b}{r}\right) - 2\cos\left(\frac{c}{r}\right)\sqrt{r^2 + r^2 \tan^2\left(\frac{a}{r}\right)}\sqrt{r^2 + r^2 \tan^2\left(\frac{b}{r}\right)}$$

And it was algebraically manipulated to simplified form:

$$c'^2 = r^2\left[2 + \tan^2\left(\frac{a}{r}\right) + \tan^2\left(\frac{b}{r}\right) - 2\sec\left(\frac{a}{r}\right)\sec\left(\frac{b}{r}\right)\cos\left(\frac{c}{r}\right)\right] \tag{41}$$

The two equations for c’ in the Law of Cosines, (40) and (41), were equated. Finally, after a series of manipulations, a new Law of Cosines was found that could be applied to spheres:

$$\cos\left(\frac{c}{r}\right) = \cos\left(\frac{a}{r}\right)\cos\left(\frac{b}{r}\right) + \sin\left(\frac{a}{r}\right)\sin\left(\frac{b}{r}\right)\cos C \tag{42}$$

<u>The Spherical Law of Sines</u>

After deriving a spherical Law of Cosines, we set out to use this new law to find a Law of Sines for spherical triangles.

We began by squaring the spherical Law of Cosines in equation (42). All of the squared cosines were able to be converted into squared sines using the following trigonometric identity

$$\cos^2\theta = 1 - \sin^2\theta \tag{43}$$

to yield

$$\sin^2 C = \frac{\sin^2\left(\frac{a}{r}\right)\sin^2\left(\frac{b}{r}\right) - 1 + \sin^2\left(\frac{c}{r}\right) + 2\cos\left(\frac{a}{r}\right)\cos\left(\frac{b}{r}\right)\cos\left(\frac{c}{r}\right) - \left[1 - \sin^2\left(\frac{a}{r}\right) - \sin^2\left(\frac{b}{r}\right) - \sin^2\left(\frac{a}{r}\right)\sin^2\left(\frac{b}{r}\right)\right]}{\sin^2\left(\frac{a}{r}\right)\sin^2\left(\frac{b}{r}\right)}$$

Through algebraic manipulation and division by $\sin^2\frac{c}{r}$, a law of sines for spherical triangles was determined:

$$\frac{\sin^2 C}{\sin^2\left(\frac{c}{r}\right)} = \frac{\sin^2\left(\frac{a}{r}\right) + \sin^2\left(\frac{b}{r}\right) + \sin^2\left(\frac{c}{r}\right) + 2\cos\left(\frac{a}{r}\right)\cos\left(\frac{b}{r}\right)\cos\left(\frac{c}{r}\right) - 2}{\sin^2\left(\frac{a}{r}\right)\sin^2\left(\frac{b}{r}\right)\sin^2\left(\frac{c}{r}\right)}$$

This new equation for the spherical Law of Sines is a symmetric function, and therefore any values within the equation can be swapped without altering the essential form of the equation. With this knowledge, we conclude that the spherical Law of Sines is

$$\frac{\sin A}{\sin\left(\frac{a}{r}\right)} = \frac{\sin B}{\sin\left(\frac{b}{r}\right)} = \frac{\sin C}{\sin\left(\frac{c}{r}\right)} \tag{44}$$

Application of Spherical Trigonometry

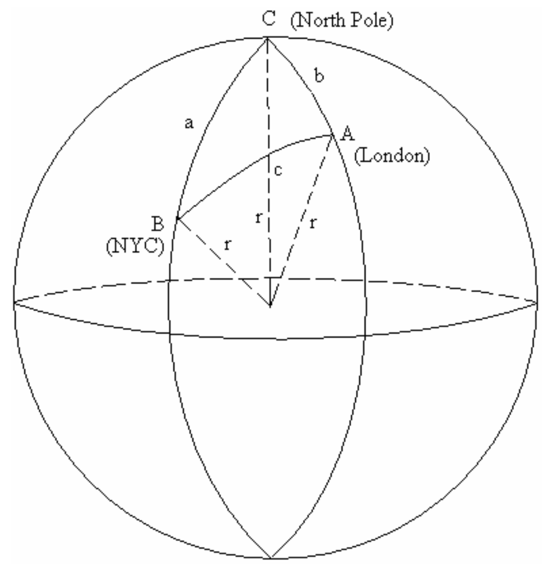

Fig. 5: Points A, B, and C connect London, New York City, and the North Pole to form a spherical triangle.

Spherical trigonometry's primary real world application involves calculating the shortest distance between two cities. Airplanes travel along the great circles of the Earth for the most efficient trip.

Given only the latitude and longitude values of London and New York City and the radius of the earth, we can solve for the distance between London and New York City using the laws that we previously derived for spherical triangles. The latitude and longitude values were converted into distances by multiplying the circumference of the earth by the degree values.

Radius of the Earth:	r = 6738.1 km [4]
Circumference:	C = 40,075.784 km
New York City:	Latitude: 40°47'N [5] = 40.738°N = 4539.9 km
	Longitude: 73°58'W [5] = 73.967°W = 8233.9 km
London:	Latitude: 51°32'N [5] = 51.533°N = 5736.6 km
	Longitude: 0°5'W [5] = 0.083°N = 9.2395 km

The spherical triangle pictured above is not useful in finding the distance between NYC and London unless *a*, *b*, and $\angle C$ are known. However, these distances can be found by sectioning off the circumference into four parts. The section in which both NYC and London lie possesses latitude values which measure the distance north of the equator; therefore, by dividing the circumference by four (10018.7) and subtracting the latitude distances, we calculated the sides that extend from the two cities to the North Pole (sides *a* and *b*). Since both longitude measurements are west of the Prime Meridian, we concluded that $\angle C$ is the difference between the degree quantities of the longitudes (73.9°).

Using the spherical Law of Cosines (42), we solved for the distance between NYC and London. The final answer that we arrived at was 5701.9 km.

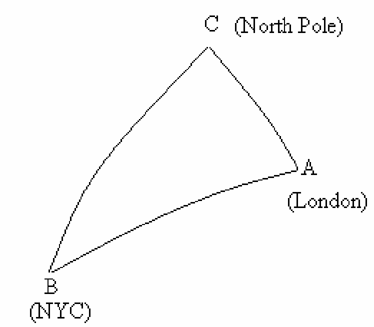

Fig. 6: Points *A*, *B*, and *C* connect London, New York City, and the North Pole to form a spherical triangle with the required values to apply the spherical law of cosines.

PART V: THE CELESTIAL SPHERE

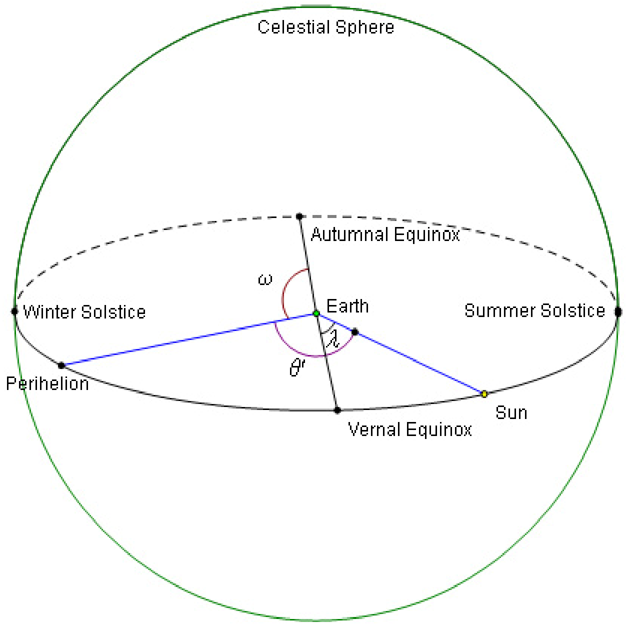

Fig. 7: Geocentric view of solar revolution.

We have been able to determine the value of θ, the angle between the sun and the Earth from a heliocentric perspective. However we have a different perspective of the stars and planets in the sky. From Earth, the celestial bodies we see in the sky lie on what is called the Celestial Sphere. The Celestial Sphere is an imaginary rotating sphere of infinite radius with Earth at its center. The Earth's poles and equator are projected onto the Celestial Sphere. If we can calculate the location of celestial bodies with respect to the celestial sphere, then we can find them in the sky.

As the Earth revolves around the sun (shown in Fig. 1), the value of θ constantly changes. In order to relate the heliocentric and geocentric perspectives, we must understand what angle θ is. In Fig. 1, θ is the angle between the Earth and its perihelion with the Sun at the vertex. In the geocentric view (Fig. 7), θ' is the angle between the Sun and its perihelion with the Earth at the vertex. Consider the situation when the Earth and the Sun are closest. (Fig. 8).

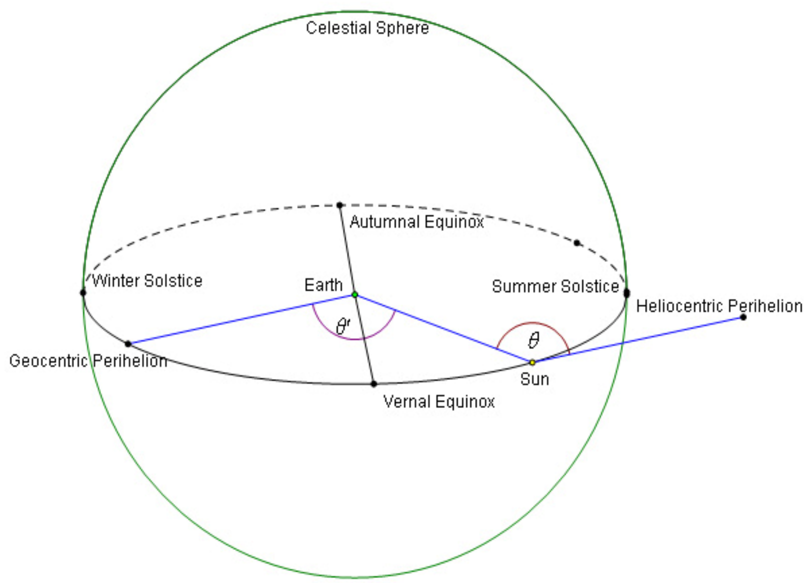

Fig. 8: Earth and Sun at perihelion.

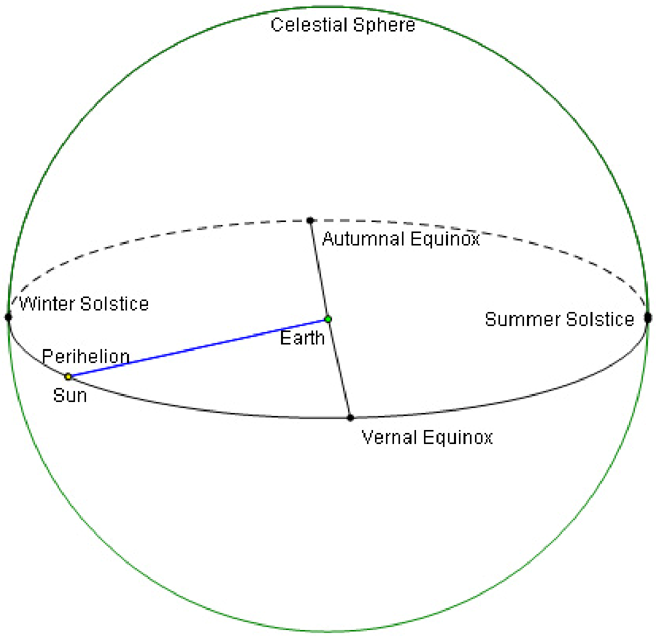

Fig. 9: Earth and Sun with translated perihelion.

At this point, the Sun and Earth are at their geocentric and heliocentric perihelia, respectively. Now allow the Sun to revolve around the Earth, as observed in the sky, while translating the perihelion (Fig. 9).

Because of parallel segments, $\theta' = \theta$. Now that we have related the heliocentric and geocentric views, we can locate celestial bodies in the sky. Referring to Fig. 7, ω is the argument of perihelion, a constant term that gives the angle between the autumnal equinox and geocentric perihelion where $\omega = 1.797$ [6]. When in the reference frame of the Celestial Sphere, it is more convenient to refer to λ instead of θ. Thus, we can calculate:

$$\lambda = \theta + \omega - \pi \qquad (45)$$

<u>The Right Ascension and Declination of the Sun</u>

Our goal in this section was to find the right ascension (α) and declination (δ) of the Sun at any given time once we calculated the value of λ. Fig. 10 represents the celestial sphere, which has a geocentric format. Therefore, we are assuming that the Sun orbits around the Earth as it seems to do in our sky. On the Celestial Sphere, the Celestial Equator is a projection of the Earth's equator onto the sky, and the North and South Celestial Poles are directly above the Earth's north and south poles, respectively. λ represents the angle the Sun forms on its yearly orbital path at any given time with respect to its position at the vernal equinox. Right ascension and declination are the coordinates of an object (in this case the Sun) on the celestial sphere. The right ascension represents the horizontal angle the Sun forms with respect to its position at the vernal equinox. It is measured in hours, minutes, and seconds with 24 hours corresponding to a full orbit. Declination represents the angle the Sun forms north or south with respect to the celestial equator and is measured in degrees (positive corresponding to north and negative to south).

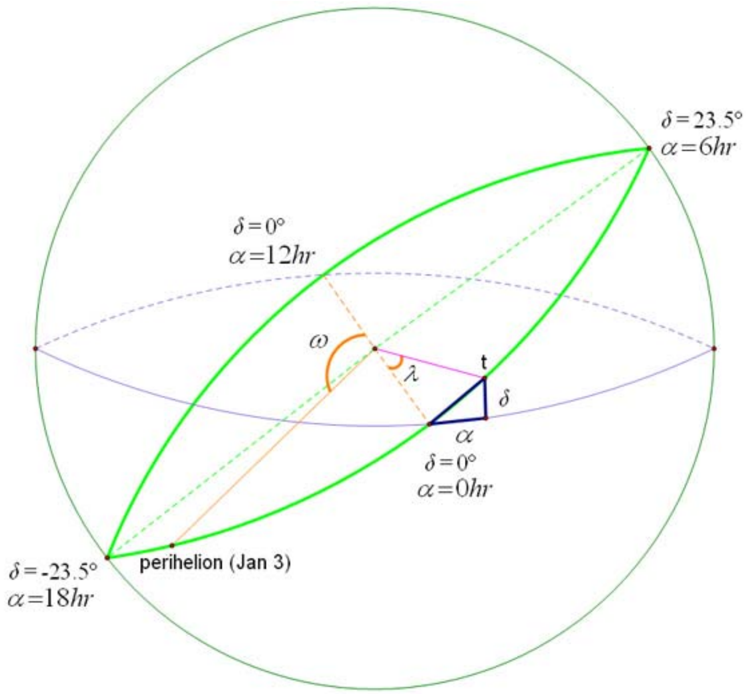

Fig. 10: Right ascension (α) and declination (δ) of the sun.

We know that ε equals 23.5° because that is the tilt of Earth's spin axis relative to its orbital path. Therefore, it is also the angle of the Sun's orbital path assuming that the earth is stationary. Also, δ measurements are by definition perpendicular to the celestial equator. All sides of the triangle formed by the right ascension, declination, and the Sun's path are parts of great circles. Therefore, we can calculate the right ascension, δ. We plugged in our proper variables into (44) to obtain:

$$\sin\delta = \sin\lambda \sin\varepsilon \tag{45}$$

We used the spherical Law of Cosines to calculate the declination. When we substituted our variables in for the spherical Law of Cosines (42), we got the formula:

$$\cos\alpha = \frac{\cos\lambda}{\cos\delta} \tag{46}$$

<u>Calculating the Time of Sunrise and Sunset</u>

By using the calculated value of δ we can determine the amount of time the sun has been in the sky at noon, and subsequently the time of sunrise and sunset. Knowing δ and right ascension for a given date, we can predict the time for sunrise and sunset. In the diagram of the Celestial Sphere (Fig. 11), φ represents the latitude of our location and the angle between our horizon and the north celestial pole, approximately 40.75°. H represents the distance the Sun travels from sunrise to noon, which can be converted to the length of time the sun has been in the sky. To perform our calculations, we need to find the value of H.

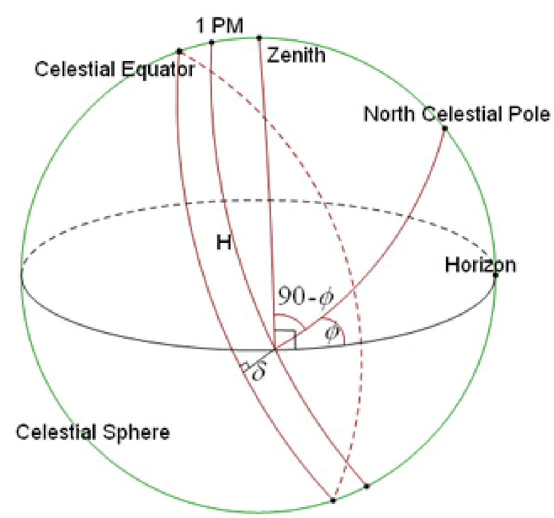

Fig. 11: The motion of the sun on the celestial sphere over the course of one day.

We can derive the arc lengths and angle measures of the triangle formed by the Earth, the zenith (the point on the Celestial Sphere intersected by a line drawn from the center of the Earth to the observer's location on the Earth's surface), and the North Celestial Pole. The angle between the zenith and the horizon is 90°, and the angle between the North Celestial Pole is φ. Therefore, the angle between the zenith and the North Celestial Pole is 90 - φ degrees. We want to solve for the angle H. It is possible to do so using the available data and spherical Law of Cosines.

$$\cos 90° = \sin(90° - \varphi)\sin(90° - \delta)\cos H + \cos(90° - \varphi)\cos(90° - \delta)$$

$$H = 106.0895379° = 7 \text{ hours } 4 \text{ minutes}$$

We now know that the sun has been in the sky for 7 hours and 4 minutes at noon on August 3, 2006. Accounting for Daylight Savings Time, we calculated that the sunrise would be at 5:56 AM and sunset at 8:04. Using the calculated value for λ for August 3, 2006 and (46), we found the declination of the sun to be 17˚15'25". Then, with (46), we calculated the right ascension to be 8h 57min 37s.

PART VI: CONSTRUCTION OF A SUNDIAL

A more practical application of celestial mechanics is the construction of a working sundial. To do this, we first place a stick of length L on the ground at an elevation of 40.75°. We must determine the angle θ that the stick's shadow casts with respect to a reference line. The reference line corresponds to the shadow cast by the sun at exactly 1PM, and by determining θ for specific times, we can construct a working sundial.

First, we realize that because the Earth rotates 360° in one day, and there are 24 hours in one day, a 15° change in the Sun's position corresponds to one hour.

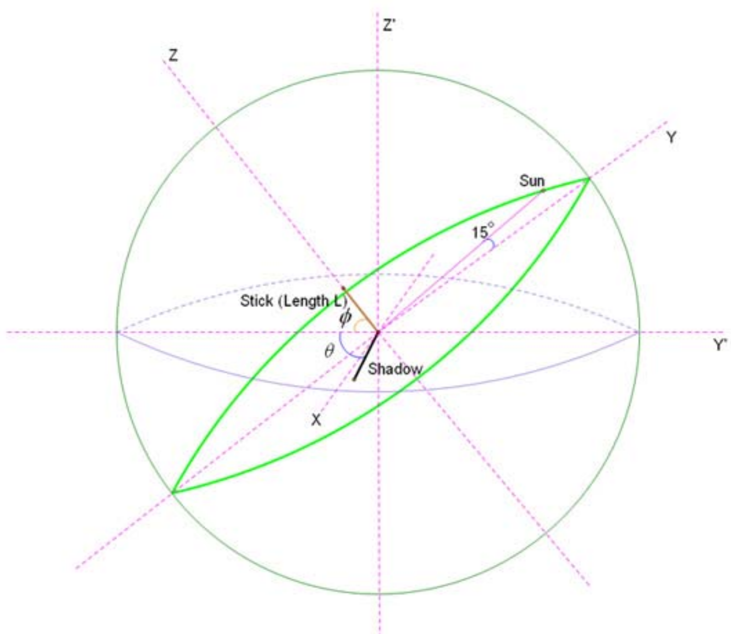

Fig. 12: A great circle describes the path of the sun across the celestial sphere.

Fig. 12 shows a tilted great circle within a sphere, with the sun lying on the great circle on a 15° angle in respect to the y-axis (this corresponds with the position of the sun at 12PM). The y-axis on the great circle corresponds to the position of the sun at 1PM, while the angle φ (which is approximately 40.75°, our latitude) corresponds to the angle of elevation of a stick of length L rising from the ground at the center of the sphere along the z-axis of the tilted axes. As previously stated, we will find the angle θ.

Because the stick lies along the z-axis, the coordinates of the stick in the plane of the greater circle is

$$(0, 0, L).$$

We draw a line of distance R from the spherical center to the sun (the height of the stick is negligible in comparison to the large value of R). The coordinates of the sun at 12PM are

$$(-R\sin 15°, R\cos 15°, 0).$$

To continue, we transform the coordinates of the sun and stick on the tilted axis to coordinates with the xy plane being the level ground, and the z axis being perpendicular to the ground. To convert these coordinates in terms of x', y', and z', we use right triangles. Note that the x-axis does not change with rotation. For the stick, we draw a right triangle.

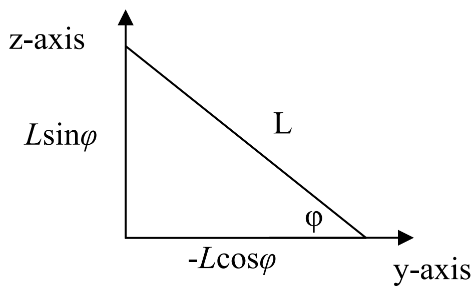

Fig. 13: Coordinates of stick after rotation of plane .

Therefore, the coordinates of the stick in the new plane are

$$(0, -L\cos\varphi, L\sin\varphi)$$

Next, we consider the new coordinates of the sun, and construct a new triangle.

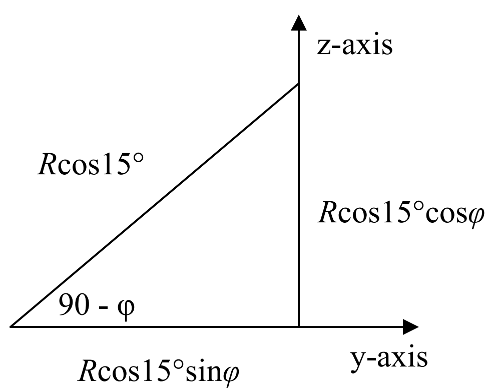

Fig. 14: Coordinates of Sun after rotation of plane.

Therefore, the coordinates of the sun in the new plane are

$$(-r\sin 15°, r\cos 15°\sin\varphi, r\cos 15°\cos\varphi).$$

In order to calculate the angle of the shadow, we must determine the equation of a line of a ray from the sun that strikes the stick. This can be determined from the two points we have already calculated for the tip of the stick and the sun:

$$x = -R\sin\eta + R\sin\eta t$$
$$y = R\cos\eta\sin\phi + (-L\cos\phi - R\cos\eta\sin\phi)t$$
$$z = R\cos\eta\cos\phi + (L\sin\phi - R\cos\eta\cos\phi)t$$

The shadow on the ground can be calculated from the extension of the line into the x'-y' plane, which is given when $z = 0$.

$$t = \frac{-R\cos\eta\cos\phi}{L\sin\phi - R\cos\eta\cos\phi}$$

Substituting t back into the equations for x and y and dividing the two gives us:

$$\frac{x}{y} = \tan\theta = \tan\eta\sin\phi$$

from which we can calculate the angles for all of the hours of the day:

Time	Θ
9:00 AM	-48.65°
10:00 AM	-33.27°
11:00 AM	-20.75°
12:00 PM	-9.97°
1:00 PM	0°
2:00 PM	9.97°
3:00 PM	20.75°
4:00 PM	33.27°
5:00 PM	48.65°

CONCLUSION

The overall objective of this project is to determine a way to use celestial mechanics to determine the positions of celestial bodies in relation to the Earth and the sun, and to also use this same knowledge for measurements on Earth, such as distance and time. Ultimately, from a basic understanding of Newton's Law and spherical geometry, we completed the objectives and generated a series of calculations that would yield information about both our planet and the other celestial bodies in our solar system.

First, from our own derivation of Kepler's Laws and an analysis of elliptical motion, we discovered an equation to calculate the position (both distance and angle) of any celestial body in its respective elliptical orbit to the sun. We furthermore derived a new Law of Sines and Law of Cosines to describe the angles and side measures of triangles on a sphere. These laws were applied to real life situations to determine the distance between the cities London and New York. More work was also done with the relationship between the earth and the celestial sphere to further pinpoint the location of the celestial bodies. Last, a sundial was successfully constructed that could effectively tell time to a high degree of accuracy, and later we successfully predicted the time of sunrise using the produced equations. Therefore, the objectives of this research project were successfully met, and an accurate system of predicting planetary motion was produced.

Celestial mechanics has ultimately proved to be a practical study through which a large amount of useful information can be derived. Now, as we look to the stars and other planets, we can see them in a new perspective, with a new understanding of how they move, and how we can predict their movements using mathematics.

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