CELESTIAL MECHANICS

Erica Che, Tyler Ernst, Minqi Jiang, Steve Kuei, Daniel Levine, Priya Marathe, Jeremy Silver, Anthony Savas, Stefan Tintea, Dmitry Vagner, Stephanie Wang

Advisor: Dr. Steve Surace
Assistant: Margaret Senese

ABSTRACT

The goal of this project was to derive the laws of physics which govern the motion of the heavenly bodies. In order to do this, Kepler's Laws of Planetary Motion were derived from simple physics equations and were then related to laws of geometry. These laws and relationships were then applied to the physical, observable universe by the use of spherical trigonometry. The celestial sphere, the imaginary sphere with the Earth at the center, was used as a model for the observable universe. All celestial objects in the night sky appear to exist on the sphere. Mathematical models can be used to predict the location of any planet on any given day. It is also possible to construct a sundial which can tell the time to a reasonable degree of accuracy.

INTRODUCTION

The heavenly bodies have been the subject of study for millennia and in that time have acquired a level of unchallengability, as stars and planets disappeared, reappeared, and moved across the night sky at regular intervals. These distant objects have had a profound effect on man's culture and development as a civilization; in the making of calendar seasons, in the navigating of the seas, and in the formation of mythology. Humans have sought to comprehend the great natural forces that drive the celestial bodies. Theories were put forth by Plato, Aristotle, Ptolemy, and Copernicus. However, the key lay in the detailed notes of Tyco Brahe; never before had anyone made so many calculations and records of the celestial bodies. Fortunately, after Brahe's death, his notes went to his young assistant, Johannes Kepler. From these records, Johannes Kepler derived what are now known as Kepler's Laws of Planetary Motion. We now recapitulate the famous analysis of Johannes Kepler and its application to the real world.

GENERAL POLAR FORMULA OF AN ELLIPSE

Using Tyco Brahe's accurate measurements, Kepler observed that the path of a celestial body appeared to be elliptical. Any ellipse with semi-major axis a, semi-minor axis b, foci at ($\pm c$, 0), and eccentricity $e = c/a$, has a general equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

If the ellipse is translated a distance c to the left so that one focus is at the origin, the formula becomes: $\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1$.

Polar coordinates in terms of an angle θ are more useful for tracking the path of an object about a fixed point. Thus, this equation, which is in rectangular form, must be converted to polar form.

Beginning with the horizontal and vertical components of a polar function: $x = r\cos\theta$ and $y = r\sin\theta$, and the definition of eccentricity: $c = ae$, we can derive the following relations of c and b in terms of the variables a and e:

$$c^2 = a^2 - b^2$$
$$b^2 = a^2 - c^2 = a^2\left(1 - e^2\right)$$

Substituting the components into the general formula gives:

$$\frac{(r\cos\theta + ae)^2}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$
$$r^2\left(a^2 + \cos^2\theta\left(-a^2e^2\right)\right) + r\left(2a^3e\left(1-e^2\right)\cos\theta\right) - a^4\left(1-e^2\right)^2 = 0$$

This is now a quadratic equation, so we solve for r:

$$r = \frac{a\left(1-e^2\right)\left(-e\cos\theta \pm 1\right)}{(1 - e\cos\theta)(1 + e\cos\theta)}$$

The radius r must be positive for the desired equation, so we take the positive root, resulting in

$$r = \frac{a\left(1-e^2\right)}{(1 + e\cos\theta)} \qquad (1)$$

Thus, (1) is the equation of an ellipse in polar form with a focus at the origin.

KEPLER'S FIRST LAW

Kepler's First Law of Planetary Motion dictates that all planets move in elliptical orbits with the Sun at one focus. To derive this law, begin with Newton's Law of Gravitation and his Second Law of Motion:

$$F = ma = -G\frac{Mm}{r^2}$$

Removing the second mass by division yields the equation for the acceleration vector as a function of orbital radius:

$$a = -G\frac{M}{r^2}$$

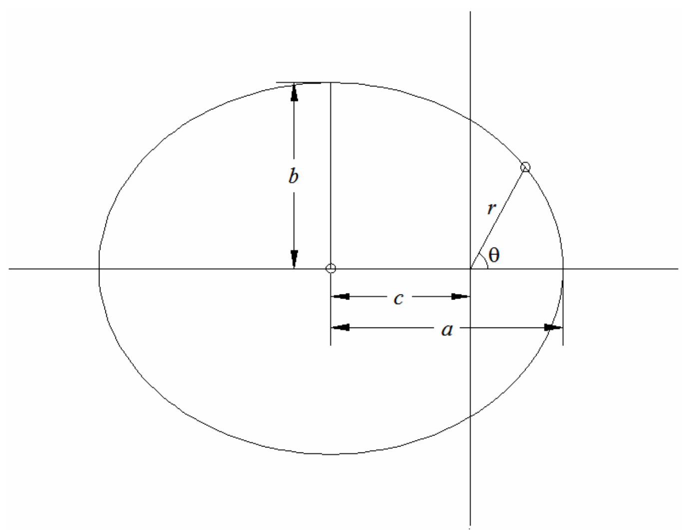

Figure 1. Ellipse with a focus (the Sun) at the origin. The r and θ in this figure represent the same quantities in the following derivation.

A general vector may be broken into its x and y components (Figure 1):

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Since acceleration is the second derivative of position, in this context, the vector components are expressed thusly:

$$\frac{d^2x}{dt^2} = -\frac{GM}{r^2}\cos\theta \qquad (2a)$$

$$\frac{d^2y}{dt^2} = -\frac{GM}{r^2}\sin\theta \qquad (2b)$$

By recursive differentiation, the vector components become:

$$\frac{d^2x}{dt^2} = \frac{d^2r}{dt^2}\cos(\theta) - 2\frac{dr}{dt}\sin(\theta)\frac{d\theta}{dt} - r\cos(\theta)\left(\frac{d\theta}{dt}\right)^2 - r\sin(\theta)\frac{d^2\theta}{dt^2}$$

$$\frac{d^2y}{dt^2} = \frac{d^2r}{dt^2}\sin(\theta) + 2\frac{dr}{dt}\cos(\theta)\frac{d\theta}{dt} - r\sin(\theta)\left(\frac{d\theta}{dt}\right)^2 + r\cos(\theta)\frac{d^2\theta}{dt^2}$$

Setting these equations equal to equations (2a) and (2b) respectively yields:

$$-\frac{GM}{r^2}\cos\theta = \frac{d^2r}{dt^2}\cos\theta - 2\frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2} \qquad (3a)$$

$$-\frac{GM}{r^2}\sin\theta = \frac{d^2r}{dt^2}\sin\theta + 2\frac{dr}{dt}\cos\theta\frac{d\theta}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2} \qquad (3b)$$

By, multiplying (3a) by cos θ and (3b) by sin θ and adding the two equations $-\frac{GM}{r^2}$ can be isolated:

$$-\frac{GM}{r^2}\cos^2\theta = \frac{d^2r}{dt^2}\cos^2\theta - 2\frac{dr}{dt}\sin\theta\cos\theta\frac{d\theta}{dt} - r\cos^2\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\cos\theta\frac{d^2\theta}{dt^2}$$

$$-\frac{GM}{r^2}\sin^2\theta = \frac{d^2r}{dt^2}\sin^2\theta + 2\frac{dr}{dt}\sin\theta\cos\theta\frac{d\theta}{dt} - r\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 + r\sin\theta\cos\theta\frac{d^2\theta}{dt^2}$$

After addition:

$$-\frac{GM}{r^2} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \qquad (4)$$

Multiplying (3a) by sin θ and (3b) by cos θ and subsequently subtracting yields:

$$-\frac{GM}{r^2}\sin(\theta)\cos(\theta) = \frac{d^2r}{dt^2}\sin(\theta)\cos(\theta) - 2\frac{dr}{dt}\sin^2(\theta)\frac{d\theta}{dt} - r\sin(\theta)\cos(\theta)\left(\frac{d\theta}{dt}\right)^2 - r\sin^2(\theta)\frac{d^2\theta}{dt^2}$$

$$-\frac{GM}{r^2}\sin(\theta)\cos(\theta) = \frac{d^2r}{dt^2}\sin(\theta)\cos(\theta) + 2\frac{dr}{dt}\cos^2(\theta)\frac{d\theta}{dt} - r\sin(\theta)\cos(\theta)\left(\frac{d\theta}{dt}\right)^2 + r\cos^2(\theta)\frac{d^2\theta}{dt^2}$$

After subtracting:

$$0 = -2\frac{dr}{dt}\frac{d\theta}{dt} - r\frac{d^2\theta}{dt^2} \qquad (5)$$

Let $p = \frac{d\theta}{dt}$ and substitute it into (5):

$$0 = -2p\frac{dr}{dt} - r\frac{dp}{dt}$$

$$-\frac{dp}{p} = 2\frac{dr}{r}$$

By integration:

$$-\ln p = 2\ln r + h$$

$$h = r^2\frac{d\theta}{dt}$$

Then substitute h, the constant of integration, into (4)

$$-\frac{GM}{r^2} = \frac{d^2r}{dt^2} - \frac{h^2}{r^3} \qquad (6)$$

Then let $u = \frac{1}{r}$.

Differentiating the resulting equation yields:

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{h}{r^2} = -r^2\frac{du}{d\theta}\cdot\frac{h}{r^2} = -h\frac{du}{d\theta}$$

$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}\frac{h}{r^2} = -h^2u^2\frac{d^2u}{d\theta^2} \qquad (7)$$

Substituting (7) into (6) yields:

$$-GMu^2 = -h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3$$

Rearranging and substituting u for r:

$$\frac{d^2u}{d\theta^2} = \frac{GM}{h^2} - u$$

By inspection:

$$u = B\cos\theta + C\sin\theta + \frac{GM}{h^2}$$

This assertion can be corroborated by differentiating twice.

Re-substituting r for u yields:

$$r = \frac{1}{B\cos\theta + C\sin\theta + \frac{GM}{h^2}} = \frac{\frac{h^2}{GM}}{\frac{h^2}{GM}\left(B\cos\theta + C\sin\theta\right) + 1} \qquad (8)$$

Requiring perihelion (the point in the orbit of closest approach) to be on the positive *x*-axis maximizes the denominator *D* of (8):

$$D = \frac{h^2}{GM}\left(B\cos\theta + C\sin\theta\right) + 1$$

$$\frac{dD}{d\theta} = \frac{h^2}{GM}\left(-B\sin\theta + C\cos\theta\right) = 0$$

$$\frac{h^2}{GM}\left(-B\sin 0 + C\cos 0\right) = \frac{h^2}{GM}(C) = 0 \Rightarrow C = 0$$

Thus the function *r* is:

$$r = \frac{\frac{h^2}{GM}}{\left(\frac{h^2}{GM}\right)B\cos\theta + 1} \qquad (9)$$

Note that *h*, *G*, and *M*, are constants (a constant of integration, the gravitational constant, and the mass of the Sun, respectively) and thus the equation matches (1), the general formula for an ellipse. Therefore the planets move in an elliptical orbit with the Sun at one focus.

KEPLER'S SECOND LAW

Kepler's Second Law states that the area swept out by a planet during its orbit in a given time period is the same for all time periods (Figure 2). In other words, the rate of change of area is constant.

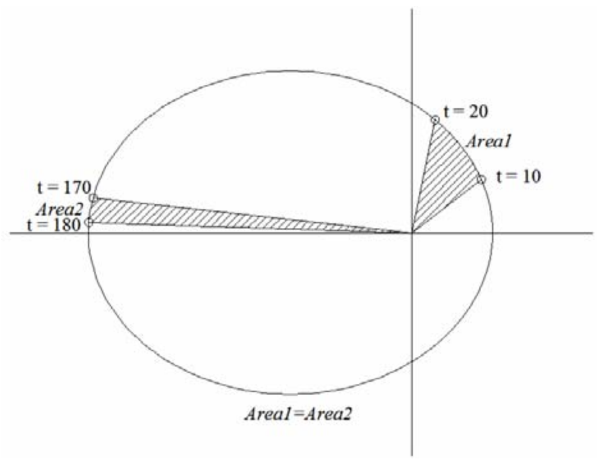

Figure 2. Kepler's Second Law

Using the laws of differentiation and integration and then substituting *h*:

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

$$\frac{dA}{d\theta} = \frac{r^2}{2}$$

$$\frac{dA}{dt} = \frac{dA}{d\theta}\cdot\frac{d\theta}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} = \frac{r^2}{2}\frac{h}{r^2} = \frac{h}{2}$$

Thus, the change in *A* over time is a constant.

KEPLER'S THIRD LAW

Kepler's 3rd Law states that the square of the period, T, of a planet's orbit is proportional to the cube of the semi-major axis, a, of that orbit.

By basic laws of integration and the fact that $\frac{dA}{dt}$ is a constant:

$$A = \int_0^T \frac{dA}{dt}dt = \frac{dA}{dt}\int_0^T dt = \frac{dA}{dt}(T-0) = \frac{dA}{dt}T$$

After rearranging:

$$T = \frac{A}{\left(\frac{dA}{dt}\right)}$$

From Kepler's 2nd Law, the derivative of the area swept out by an orbiting body with respect to time is $\frac{h}{2}$.

Thus,

$$T = \frac{2A}{h}$$

Substituting πab for A:

$$T = \frac{2\pi ab}{h} \qquad (10)$$

This equation is then squared because Kepler's Third Law deals with the square of the period.

$$T^2 = \frac{4\pi^2 a^2 b^2}{h^2}$$

Using the relationships between a, b, c, and e (the elliptical constants):

$$T^2 = \frac{4\pi^2 a^4 (1-e^2)}{h^2}$$

As can be seen in Figure 1, the maximum value of the orbital radius is at $r = a + c$ and the minimum value is at $r = a - c$.
Thus,

$$a - c \leq \frac{\frac{h^2}{GM}}{\frac{Bh^2}{GM}\cos\theta + 1} \leq a + c$$

This equation which employs (9) is then used to solve for h^2 at both the minimum and maximum

values of r, when $\cos\theta = -1$ and 1, respectively.
This yields:

$$h^2_{r\,\min} = \frac{GM(a-c)}{1-B(a-c)}$$

and

$$h^2_{r\,\max} = \frac{GM(a+c)}{1+B(a+c)}$$

Both values of h^2 are substituted into the equation for T^2. These equations are then added and divided by two. This allows the constant B to cancel from both equations. Again using the elliptical constant relationships, the equation is simplified to yield

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

This relationship between T^2 and a^3 is the final equation that Newton derived from Kepler's general relationship between T^2 and a^3 of a planet's orbit, and shows that the two are, in fact, related.

APPLYING KEPLER'S LAWS TO THE OBSERVABLE UNIVERSE

Relating an Ellipse to its Circumscribing Circle

It turns out to be extremely difficult to track a planet's orbit by using angle θ alone, so a new and more easily traceable angle E is found by inscribing the ellipse in a circle. The new angle E is formed from the center of the circle to the point on the circle formed by the intersection of the circle and the perpendicular line extending from the x-axis through the planets position on the ellipse (Figure 3). A useful relationship between θ and E is determined from Figure 3 using simple right triangle trigonometry:

$$\cos \mathrm{E} = \frac{c + r\cos\theta}{a}$$

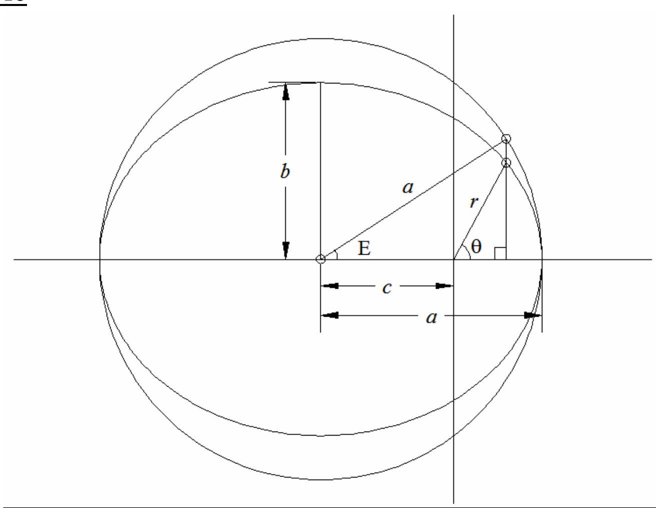

Figure 3. The circle circumscribed around the elliptical orbit.

Then substituting $c = ae$ and simplifying,

$$a\cos \mathrm{E} = ae + \left[\frac{a(1-e^2)}{1+e\cos\theta}\right]\cos\theta$$

$$\cos \mathrm{E} = \frac{e+\cos\theta}{1+e\cos\theta} \qquad (11)$$

Then, since this relation is much more useful in terms of tangents and half-angles, it is transformed using the formula for the half-angle of a tangent:

$$\tan\frac{\mathrm{E}}{2} = \frac{\sin \mathrm{E}}{1+\cos \mathrm{E}}$$

$$\tan\frac{\mathrm{E}}{2} = \frac{\sin \mathrm{E}}{1+\dfrac{e+\cos\theta}{1+e\cos\theta}}$$

By constructing a right triangle with the two side lengths used to determine the cosine (the adjacent and hypotenuse), and then solving for the third (opposite) side, one finds that:

$$\sin \mathrm{E} = \frac{\sqrt{(1-e^2)(\sin^2\theta)}}{1+e\cos\theta}$$

$$\tan\frac{\mathrm{E}}{2} = \frac{\dfrac{\sin\theta\sqrt{(1-e^2)}}{1+e\cos\theta}}{\dfrac{(1+e)(1+\cos\theta)}{1+e\cos\theta}}$$

$$\tan\frac{\mathrm{E}}{2} = \frac{\sin\theta}{1+\cos\theta}\left(\frac{\sqrt{(1-e^2)}}{1+e}\right)$$

$$\tan\frac{\mathrm{E}}{2} = \tan\frac{\theta}{2}\left(\frac{\sqrt{(1+e)(1-e)}}{1+e}\right)\left(\frac{\sqrt{(1+e)}}{\sqrt{(1+e)}}\right)$$

$$\tan\frac{\mathrm{E}}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}$$

Kepler's Laws and Elliptical Geometry

The final step in creating a practical application for Kepler's Laws is to determine the position of a planet with respect to time.

Beginning with Kepler's Equation, which includes some new quantity *M*:

$$M = E - e\sin E \qquad (12)$$

Thus,

$$\frac{dM}{dt} = \frac{dE}{dt} - e\cos E\frac{dE}{dt} = \frac{dE}{dt}\left(1-e\cos E\right) \qquad (13)$$

Differentiating and rearranging equation (11) gives:

$$\frac{dE}{dt} = \frac{(1-e^2)\sin\theta \dfrac{d\theta}{dt}}{(1+e\cos\theta)^2 \sin E} \qquad (14)$$

The ratio of the y-coordinates at a given value of x for a circumscribing circle and an ellipse is given by $\dfrac{y_{circle}}{y_{ellipse}} = \dfrac{a}{b}$

Using Figure 3 and right triangle trigonometry:

$$y_{ellipse} = r\sin\theta$$

$$\frac{y_{circle}}{r\sin\theta} = \frac{a}{b}$$

$$y_{circle} = \frac{ar\sin\theta}{b}$$

$$\sin E = \frac{y_{circle}}{a}$$

$$\sin E = \frac{r\sin\theta}{b} \qquad (15)$$

Substituting equation (15) into (14) and (1) in for r gives:

$$\frac{dE}{dt} = \frac{b\dfrac{d\theta}{dt}}{a(1+e\cos\theta)}$$

Replacing $\dfrac{d\theta}{dt}$ with $\dfrac{h}{r^2}$ and again using (1) for r further simplifies to:

$$\frac{dE}{dt} = \frac{hb(1+e\cos\theta)}{a^3(1-e^2)^2} \qquad (16)$$

Next, substitute (16) and (11) into (13) to produce:

$$\frac{dM}{dt} = \frac{h}{ab} \qquad (17)$$

Rearranging (10) to solve for h in terms of T, a, and b and substituting it into (17) gives the result:

$$\frac{dM}{dt} = \frac{2\pi}{T} = \text{constant}$$

Now integrating,

$$M = \frac{2\pi\, t}{T} + C$$

Since M (0) = 0, the constant of integration C = 0.

$$M = \frac{2\pi\, t}{T}$$

Real World Application

The next goal is to apply these equations to the physical universe by determining the values for M (commonly called the "mean anomaly"), E (commonly called the "eccentric anomaly"), θ, and r.

For the planet Earth, T = 365.2425 days [1], $a = 1.496 \cdot 10^{11}$ m [2], and e = 0.0167 [3]. The date of the perihelion in 2007 was January 3, 20:00 Universal Coordinated Time [4] or January 3, 3:00 P.M. local time.

On July 25, 2007, t = 203 days (as counted from the perihelion date).

$$M = \frac{2\pi(203)}{365.25} = 3.492 \text{ radians}$$

Using e, this value of M, the fact that E is in the third quadrant on July 25, and (12) gives $E = 3.489$ radians.

Using this value of E and (11), we find θ to be: 3.484 radians.

(Note: the values for M, E, and θ are very similar because the Earth's elliptical orbit is nearly circular, e = 0.0167.)

Finally, we use equation (1) to find $r = 1.519 \cdot 10^{11}$ m.

SPHERICAL TRIGONOMETRY

Spherical trigonometry is a fundamental tool in the development of the relationships involved in celestial mechanics. Spherical trigonometry differs from planar trigonometry on many fundamental properties. Using basic properties of spherical angles and triangles, the spherical Law of Sines and spherical Law of Cosines were derived. These relations were then crucial tools in deriving relationships to track the motion of celestial bodies.

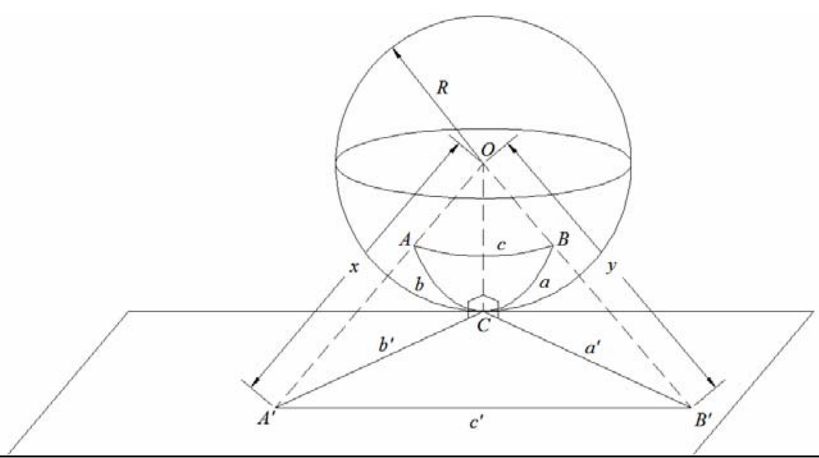

Figure 4. A spherical triangle projected onto a plane yields a planar triangle.

Spherical Law of Cosines

To derive a relationship between the sides and angles of spherical triangles one can use known relationships about planar triangles, including the Pythagorean Theorem, Law of Cosines, and Law of Sines. One must take a sphere with a triangle on the surface and place the sphere on a plane, tangent to one of the vertices (in this case, point *C*) of the spherical triangle. One can then extend the line between the center of the sphere and point *A* until it intersects the plane at point A'. Doing the same for point *B*, there now exists a planar triangle $A'B'C$. By extending the radius *OC* there now exist two right triangles OCA' and OCB' (Figure 4). Using the four planar triangles OCA', OCB', $OA'B'$, and $A'B'C$, one can use the aforementioned theorems and laws to determine relationships between the sides in terms of the spherical variables *A*, *B*, *C*, *a*, *b*, *c*, and *R*.

The angles of the triangles can be redefined in terms of the arc they sweep out as:

$$S = R\theta$$

where *S* is the arc length, *R* is the radius, and θ is the angle that sweeps out the arc.

Thus:

$$\angle A'OC = \frac{b}{R}$$

$$\angle B'OC = \frac{a}{R}$$

The tangents of OCB' and OCA', respectively were used to find:

$$a' = R\tan\frac{a}{R} \qquad (18a)$$

$$b' = R\tan\frac{b}{R} \qquad (18b)$$

The Law of Cosines as applied to triangle $A'B'C$ is:

$$c'^2 = a'^2 + b'^2 - 2a'b'\cos C$$

After substituting (18*a*) and (18*b*) and rearranging:

$$c' = R\sqrt{\tan^2\frac{a}{R} + \tan^2\frac{b}{R} - 2\tan\frac{a}{R}\tan\frac{b}{R}\cos C}$$

Hypotenuses of the right triangles OCA' and OCB', respectively:

$$x = \sqrt{R^2 + b'^2} = \sqrt{R^2\left(1 + \tan^2\frac{b}{R}\right)} = R\sec\frac{b}{R} \qquad (19a)$$

$$y = \sqrt{R^2 + a'^2} = \sqrt{R^2\left(1 + \tan^2\frac{a}{R}\right)} = R\sec\frac{a}{R} \qquad (19b)$$

The Law of Cosines as applied to triangle $\Delta OA'B'$ is:

$$c'^2 = x^2 + y^2 - 2xy\cos\frac{c}{R}$$

After substituting (19*a*) and (19*b*) and rearranging:

$$c' = R\sqrt{\sec^2\frac{a}{R} + \sec^2\frac{b}{R} - 2\sec\frac{a}{R}\sec\frac{b}{R}\cos\frac{c}{R}}$$

One can equate the two expressions for c' and use the identity, $1+\tan^2\theta = \sec^2\theta$, to yield:

$$\tan^2\frac{a}{R} + \tan^2\frac{b}{R} - 2\tan\frac{a}{R}\tan\frac{b}{R}\cos C = 1 + \tan^2\frac{a}{R} + 1 + \tan^2\frac{b}{R} - 2\sec\frac{a}{R}\sec\frac{b}{R}\cos\frac{c}{R}$$

From simple algebraic manipulation one gets:

$$\cos\frac{c}{R} = \frac{1+\tan\frac{a}{R}\tan\frac{b}{R}\cos C}{\sec\frac{a}{R}\sec\frac{b}{R}}$$

After rewriting all trigonometric functions in terms of sine and cosine:

$$\cos\frac{c}{R} = \frac{1+\left(\dfrac{\sin\frac{a}{R}\sin\frac{b}{R}}{\cos\frac{a}{R}\cos\frac{b}{R}}\right)\cos C}{\left(\dfrac{1}{\cos\frac{a}{R}\cos\frac{b}{R}}\right)}$$

Simplifying this, one gets the spherical analogue for the Law of Cosines

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cos\frac{b}{R} + \sin\frac{a}{R}\sin\frac{b}{R}\cos C \qquad (20)$$

<u>Spherical Law of Sines</u>

Once one has the equation for the Law of Cosines, one can derive sin *C* from cos *C* using the identity $\cos^2\theta + \sin^2\theta = 1$.

Starting with (20) after algebraic manipulation:

$$\cos C = \frac{\cos\frac{c}{R} - \cos\frac{a}{R}\cos\frac{b}{R}}{\sin\frac{a}{R}\sin\frac{b}{R}}$$

Using the Pythagorean Identity $\sin\theta = \sqrt{1-\cos^2\theta}$ to replace cos C with sin C, yields:

$$\sin C = \sqrt{1-\cos^2 C} = \sqrt{1 - \frac{\cos^2\frac{c}{R} - 2\cos\frac{a}{R}\cos\frac{b}{R}\cos\frac{c}{R} + \cos^2\frac{a}{R}\cos^2\frac{b}{R}}{\sin^2\frac{a}{R}\sin^2\frac{b}{R}}}$$

After squaring both sides and simplifying:

$$\sin^2 C = \frac{\sin^2\frac{a}{R}\sin^2\frac{b}{R} - \cos^2\frac{c}{R} + 2\cos\frac{a}{R}\cos\frac{b}{R}\cos\frac{c}{R} - \cos^2\frac{a}{R}\cos^2\frac{b}{R}}{\sin^2\frac{a}{R}\sin^2\frac{b}{R}}$$

After further substitution and simplification:

$$\sin^2 C = \frac{\sin^2\frac{a}{R} + \sin^2\frac{b}{R} + \sin^2\frac{c}{R} + 2\cos\frac{a}{R}\cos\frac{b}{R}\cos\frac{c}{R}}{\sin^2\frac{a}{R}\sin^2\frac{b}{R}}$$

Dividing by $\sin^2\frac{c}{R}$ yields:

$$\frac{\sin^2 C}{\sin^2\frac{c}{R}} = \frac{\sin^2\frac{a}{R} + \sin^2\frac{b}{R} + \sin^2\frac{c}{R} + 2\cos\frac{a}{R}\cos\frac{b}{R}\cos\frac{c}{R}}{\sin^2\frac{a}{R}\sin^2\frac{b}{R}\sin^2\frac{c}{R}}$$

The right side of the equation is symmetric in a, b, and c, therefore it is a constant for a given triangle and one can equate the three terms as:

$$\frac{\sin^2 A}{\sin^2\frac{a}{R}} = \frac{\sin^2 B}{\sin^2\frac{b}{R}} = \frac{\sin^2 C}{\sin^2\frac{c}{R}}$$

Further simplification yields the Spherical Law of Sines:

$$\frac{\sin A}{\sin\frac{a}{R}} = \frac{\sin B}{\sin\frac{b}{R}} = \frac{\sin C}{\sin\frac{c}{R}} \qquad (21)$$

Practical Application

Distance Between Two Cities

An application of the Spherical Law of Cosines is finding the shortest distance between two cities on the Earth given their latitude and longitude coordinates. One can create a spherical triangle with vertices at the two cities and a pole (Figure 5). The angular arc length of the side between a given city and the pole is the complement of the latitude. This gives the value of two adjacent sides. The angle between them is the difference in the longitudes of the two cities. Given two adjacent sides and an included angle, one can find the third side, the distance between the cities, using the Spherical Law of Cosines. This will yield the angular arc length; the linear distance can be found by multiplying the angle by the radius of the Earth.

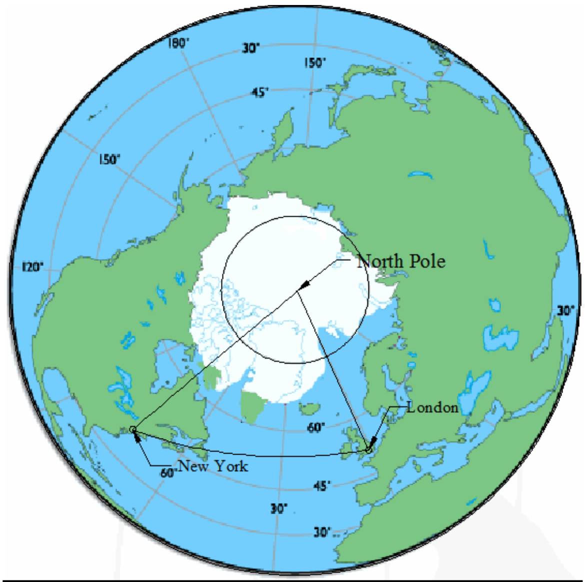

Figure 5. The shortest distance between two cities is the great circle connecting them

An example would be finding the distance between New York City and London. The coordinates of New York are 40°40'N and 73°56'W, and those of London are 51°30'N and 0°07'W. The difference in longitudes would be 74.05°. When the triangle is created as above (Figure 5), one gets one side to be 49°20', another side to be 39°30', and the included angle to be 73°49'. Using (20), one gets the included side to be .8834 radians, and when multiplied by the radius of the Earth (3963 miles) the distance comes out to be 3501 miles.

Relating θ, λ, and ω

The Universal Reference

As the orbits of the planets are different, one must use a universal reference for any calculations that need to be done. The Universal Reference Line is the line passing through the Sun and pointing to the constellation Aries. Lambda (λ) is defined as the angle between the Universal Reference and the line connecting the centers of the Earth and the Sun at any given time. Theta (θ) is defined as the counterclockwise angle from the perihelion to the Earth with the Sun as the vertex. In

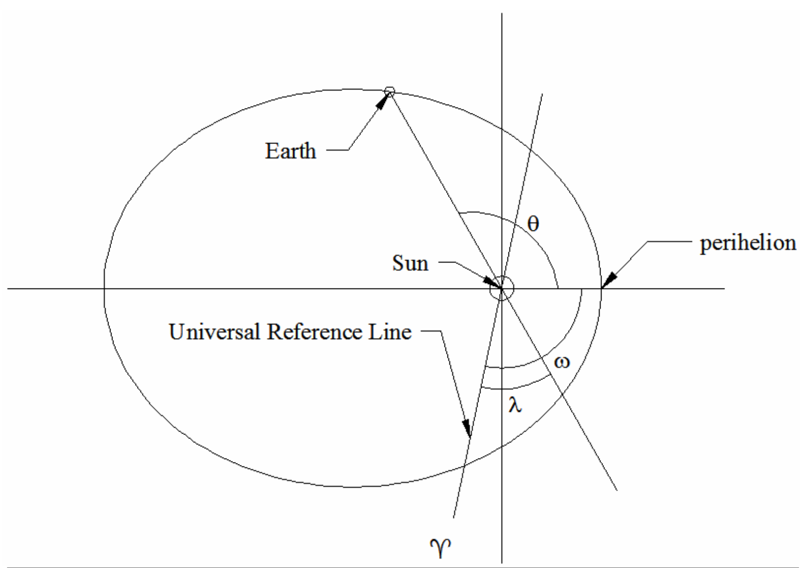

Figure 6. Diagram of the Universal Reference Line with respect to Earth and its orbit

addition, omega (ω) is defined as the angle between the Universal Reference and the perihelion (Figure 6).

Observing the relationship between λ, ω, and θ, one finds that:

$$\lambda = \theta + \omega - \pi$$

The Celestial Sphere

For the purposes of astronomical calculations the universe can be modeled as celestial bodies on the surface of an enormous celestial sphere centered at the Earth. The celestial poles are aligned with the North and South Poles of the Earth. Similarly, the celestial equator is a great circle concentric with the equator of the Earth. Points on the celestial sphere may be located in terms of two quantities: right ascension, α, and declination, δ. These two are the celestial equivalents of terrestrial longitude and latitude respectively. However, unlike longitude, the right ascension is measured in terms of hours, with 24 hours equivalent to 360 degrees. This angle is measured counterclockwise with respect to the Universal Reference Line. Furthermore, arcs on the celestial sphere are denoted by the central angle that sweeps out that arc.

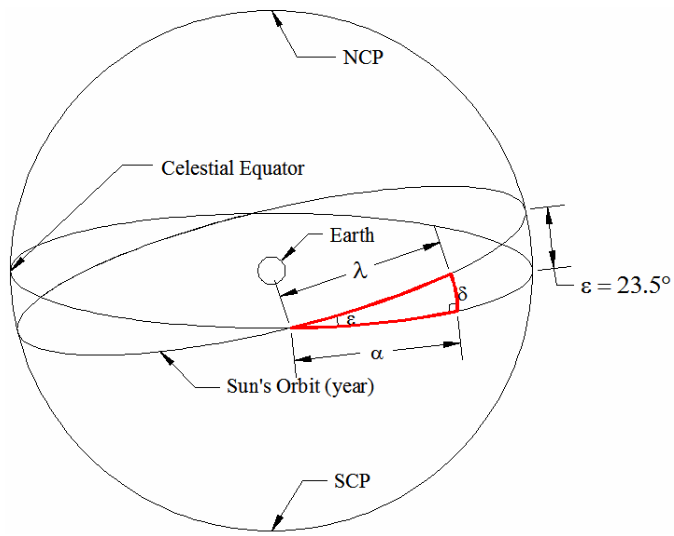

Figure 7. The Celestial Sphere

Throughout the course of a year, the Sun travels on a great circle, known as the ecliptic, with maximum declination ε=23.5. This maximum declination is equivalent to the Tropic of Cancer on Earth. One can also write an equation relating declination δ to λ and ε.

Applying (21) to the highlighted spherical triangle in Figure 7, one finds that:

$$\frac{\sin\varepsilon}{\sin\delta} = \frac{\sin\left(\dfrac{\pi}{2}\right)}{\sin\lambda}$$

After simplifying:

$$\sin\delta = \sin\varepsilon\sin\lambda$$

Applying (20) to the highlighted triangle, one finds that

$$\cos\lambda = \cos\alpha\cos\delta + \sin\alpha\sin\delta\cos\left(\frac{\pi}{2}\right)$$

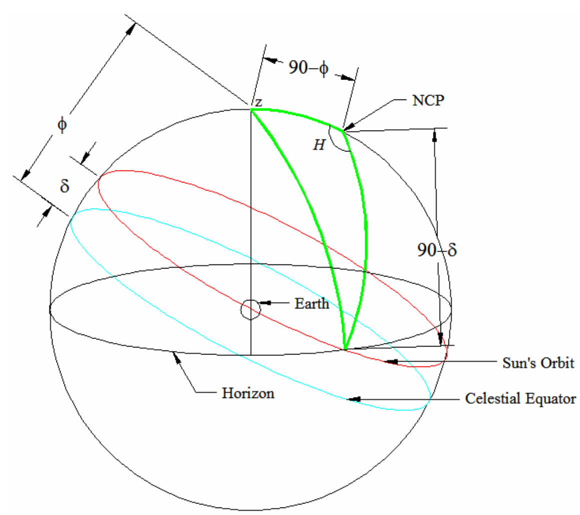

Figure 8. Diagram of Sunrise with respect to the celestial sphere

After simplifying:

$$\cos\alpha = \frac{\cos\lambda}{\cos\delta}$$

Practical Application

Sunrise Times

Using these spherical trigonometric laws on the Celestial Sphere, one can determine the approximate time of sunrise. Let ϕ be the latitude of a terrestrial observer. Using the spherical triangle that is highlighted in Figure 8, one can use the Spherical Law of Cosines to set up a relationship among ϕ, δ, and H, the angular arc length between the Sun's highest point in the sky and sunrise:

$$\cos\frac{\pi}{2} = \cos\left(\frac{\pi}{2}-\phi\right)\cos\left(\frac{\pi}{2}-\delta\right)+\sin\left(\frac{\pi}{2}-\phi\right)\sin\left(\frac{\pi}{2}-\delta\right)\cos H \qquad (22)$$

Using the fact that cosine and sine are out of phase by $\frac{\pi}{2}$, (22) can be rewritten:

$$0 = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H$$

After simplification:

$$\cos H = -\tan\phi\tan\delta \qquad (23)$$

In order to test the model for calculating sunrise times, the sunrise time was determined for a particular day, July 30. The declination of the Sun for this day was 18°40' [7], and the latitude of the location where the sunrise was recorded was 41°. Using the values for declination δ and latitude ϕ in (23) we found a value for H. H is the angular measure between the Sun's highest point in the sky and sunrise. The Sun's highest point in the sky is normally noon, but July 29 falls during the daylight savings time period so the Sun's highest point occurs at 1:00 P.M. instead. This value is an angular measure that must be converted into hours. The conversion factor is 360° = 24 hours. Once converted, the value must be subtracted from 1:00 P.M.

$$\cos H = -\tan\phi\tan\delta$$
$$\cos H = -\tan 41^\circ \tan 18^\circ 40'$$
$$H = \cos^{-1}\left[-\tan 41^\circ \tan 18^\circ 40'\right]$$
$$H \approx 107.078^\circ$$

Converting this into hours, one gets:

$$H \approx 7.139 \text{ hours}$$
$$H \approx 7 \text{ hours, 8 minutes, 20 seconds}$$

After subtracting this from 1:00 P.M. the approximate sunrise time is 5:51 A.M.

CONSTRUCTING A SUNDIAL

In order to construct an accurate sundial, the angle of the shadow cast by a stick every hour had to be calculated. These calculations were performed using a model of a dome to represent the sky in the vicinity of the stick and the stick tilted at an angle of φ (equal in value to the latitude at which the stick is constructed) towards the north (Figure 9).

In order to be able to trace the sunrays' paths to the ground and determine where the shadow of the stick will be at each hour, the diagram was plotted onto a coordinate system, with the *x* and *y* axes along the plane of the Sun's path, and the *z* axis along the stick. The Sun rises and travels across the sky, reaching its highest point at 1:00 P.M. due to Daylight Savings Time. At this time, the stick's shadow points directly north. It is given that the Sun travels 15° in the sky each hour. Thus, at 12:00 PM, the Sun is 15° to the east of its highest point. The coordinates of the Sun (s) and the tip of the stick (T) at this time were determined.

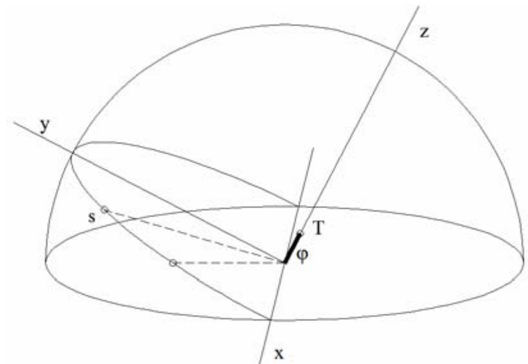

Figure 9. The Sundial with respect to the sky in its vicinity. s represents the Sun.

The coordinates with respect to the original axes:

$$T:(0, 0, L)$$
$$s:(r\sin 15°, r\cos 15°, 0)$$

Afterwards, the entire coordinate plane was rotated so that the *x* and *y* axes lay along the plane of the ground (Figure 10).

The coordinates with respect to the rotated axes:

$$T:(0, -L\cos\varphi, L\sin\varphi)$$
$$s:(r\sin 15°, r\cos 15°\sin\varphi, r\cos 15°\cos\varphi)$$

The equation of the line that connects the two points s and T in three-dimensional space was determined and the point at which it intersected the *x–y* coordinate plane (also known as the ground) was calculated in order to find the angle θ of the stick's shadow at 12:00 P.M. Note that time t is measured in hours.

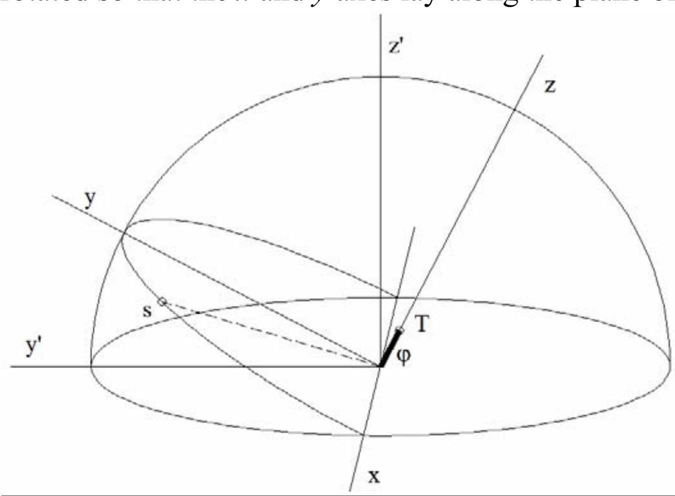

Figure 10 . The Sundial and its vicinity with rotated coordinate axes

The line is found to be:

$$x = tr\sin 15°$$
$$y = -L\cos\varphi + t(r\cos 15°\sin\varphi + L\cos\varphi)$$
$$z = L\sin\varphi + t(r\cos 15°\cos\varphi - L\sin\varphi)$$

If it intersects the *x-y* plane, z = 0, therefore,

$$t = \frac{-L\sin\varphi}{r\cos 15^\circ \cos\varphi - L\sin\varphi}$$

Substituting t,

$$x = \frac{-L\sin\varphi r\sin 15^\circ}{r\cos 15^\circ \cos\varphi - L\sin\varphi}$$

$$y = \frac{-L\cos^2\varphi r\cos 15^\circ + L^2\cos\varphi\sin\varphi - Lr\cos 15^\circ \sin^2\varphi - L^2\sin\varphi\cos\varphi}{r\cos 15^\circ \cos\varphi - L\sin\varphi}$$

Divide $\frac{x}{y} = \tan\theta$

$$\tan\theta = \frac{\text{- Lsin}\varphi r\sin 15^\circ}{-Lr\cos^2\varphi\cos 15^\circ - Lr\cos 15^\circ \sin^2\varphi} = \frac{-Lr\sin\varphi\sin 15^\circ}{-Lr\cos 15^\circ(\cos^2\varphi + \sin^2\varphi)}$$

$$\tan\theta = \sin\varphi\tan 15^\circ$$

This becomes the general equation to find the angle of the stick's shadow at any hour of the day:

$$\tan\theta_{\text{shadow}} = \sin\varphi\tan\theta_{\text{Sun.}}$$

The following table (Table 1) was calculated for every hour with $\varphi = 41^\circ$:

Table 1: Sundial Angles During Daylight Hours

Time	θ_{Sun}	θ_{shadow}
1 PM	0°	0°
2 PM/12 PM	15°	9.97°
3 PM/11 AM	30°	20.75°
4 PM/10 AM	45°	33.27°
5 PM/9 AM	60°	48.65°
6 PM/8AM	75°	67.78°
7 PM/7AM	90°	90°

CONCLUSION

The purpose of this research was to explore the motion of heavenly bodies and apply mathematical and physical formulae to track their orbits. Kepler's Laws became the foundation of this project, as they explain the fundamentals of planetary orbits. These laws were derived using calculus. Elliptical geometry was then used to confirm that these laws were, in fact, Kepler's Laws. Afterwards, a new Law of Sines and Law of Cosines were derived for use on spherical triangles, so that planetary orbits could be viewed as they are from our earthly perspective. With these findings, several related problems can be solved. The Law of Cosines was used to determine sunrise time, as well as the distance between two cities. Spherical and Cartesian geometry was also used to construct a sundial that now accurately tells the time of day, once again confirming Kepler's Laws.

REFERENCES

[1] Simon Cassidy. The Tropical and the Anomalistic Year. 1996. <http://www.hermetic.ch/cal_stud/cassidy/tropanom.htm> Accessed 2007 Jul 25.

[2] Encyclopædia Britannica. <http://www.britannica.com/eb/article-9010001/astronomical-unit> Accessed 2007 Jul 25.

[3] [UCAR] University Corporation for Atmospheric Research. 2005 December 16. University of Michigan. <http://www.windows.ucar.edu/tour/link=/physical_science/physics/mechanics/orbit/eccentricity.html&edu=high>. Accessed 2007 Jul 25.

[4] U.S. naval observatory webmaster. 2003 Oct 30. Earth's seasons, etc. <http://aa.usno.navy.mil/data/docs/EarthSeasons.html>. Accessed 2007 Jul 25.

[5] Weather Day Data. 2003 Oct 17. <http://nschubert.home.mchsi.com/education/ssf/october/owddata.html>. Accessed 2007 Jul 29.

[6] Current local time in London. <http://www.timeanddate.com/worldclock/city.html?n=136>. Accessed 2007 Jul 29.

[7] Table of the Declination of the Sun. <http://www.wsanford.com/~wsanford/exo/sundials/DEC_Sun.html>. Accessed 2007 Jul 29.